Notations, Conventions, and Constraints: Contributions to Effective Uses of Concrete Materials in Elementary Mathematics

Patrick W. Thompson

San Diego State University

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Abstract

Twenty 4th-grade children were matched according to performance on a whole-number calculation and concepts pretest and assigned at random to one of two groups: wooden base-ten blocks and computerized microworld. Instruction in each group was designed to orient students toward relationships between notation and meaning. Instruction given the two groups was based upon a single script that extended whole number numeration to decimal numeration, and emphasized solving problems in concrete settings while inventing notational schemes to represent steps in solutions. Neither group changed in regard to whole-number notational methods. Blocks children understood decimal numerals as if they were funny whole numbers; Microworld children attempted to give meaning to decimal notational methods, but were largely in a state of disequilibrium at the end of the study.

The use of concrete materials in elementary mathematics instruction has been widely advocated. Textbooks on methods for teaching elementary school mathematics and college textbooks on mathematics for elementary school teachers promote liberal use of concrete materials. Yet, the research literature on effectiveness of instruction involving uses of concrete materials is equivocal at best (Fennema, 1972; Labinowicz, 1985; Resnick, 1982; Resnick & Omanson, 1987; Sowell, 1989; Suydam & Higgins, 1977; Wearne & Hiebert, 1988).

The equivocacy of past research on the beneficial effects of concrete materials is illustrated by comparing studies by Resnick and Omanson (1987), Fuson and Briars (1990), Labinowicz (1985), and Wearne and Hiebert (1988). In Resnick and Omanson's (1987) study, the use of base-ten blocks had little impact on children's understanding of and skill with multi-digit place-value subtraction. Fuson and Briars (1990) reported that children achieved a high level of skill with multi-digit addition and subtraction through the use of base-ten blocks. Labinowicz (1985) reported that third-graders using base-ten blocks developed relatively little computation skill. Wearne and Hiebert (1988) reported that the use of base-ten blocks had a perceptible impact on fourth, fifth, and sixth graders' development of meaning for decimal numeration and for symbolic addition and subtraction with decimal fractions. Some of the equivocacy can be attributed to the goals by which the studies were conducted and evaluated (computational facility versus problem solving) or to their orientation toward algorithms (prescription versus invention). However, not all the differences in results can be attributed to these sources. The nature of students' engagement with concrete materials and their orientation toward materials in relation to notation and numerical value is another possibility (Fuson & Briars, 1990).

We can place students' use of concrete materials in a larger context—their development of an orientation toward using notation to express their reasoning as it occurs in concrete settings. In this regard we must be sensitive to students' image of their activities. Resnick and Omanson (1987) observed that students' active participation in a prescribed activity may have little effect if students think that they are following a prescription. Students' reenactment of a prescribed procedure does not give them opportunities to construct constraints in their meanings and reasoning—they meet constraints only because they are obliged to adhere to prescription, and it matters little that the prescriptions entail use of concrete materials.

In reenacting prescribed procedures, students do not experience constraints as arising from tensions between their attempts to say what they have in mind and their attempts to be systematic in their expressions of it. As students come to be systematic in their expressions of reasoning and make a commitment to express their reasoning within their system, that same systematicity places constraints on the reasoning they wish to express. When students are aware of the constraining influence exerted by their arbitrary use of notation, they may feel freer to modify their standard uses of notation to express better what they have in mind. Also, when students are aware of reciprocal relationships between notation and reasoning they may be more inclined to concentrate on their reasoning when experiencing difficulty and concentrate less on performing correct notational actions. A person's meaningful use of notation can be highly idiosyncratic, it can be creative expression constrained by convention, or it can be an automated use of convention. In the first case the individual is engaging in personal expression. In the second case the individual is conforming to convention with the awareness of conforming. In the third case the individual is using convention unthinkingly—perhaps unknowingly.

Conventions, Curriculum, and Research

One way to approach notational and methodological convention in school mathematics is to allow students to create their own problem-solving methods in concrete settings and to treat notational expressions of method as a mathematical problem. One aim of this approach is for students to understand that any notational method is but one of many valid ways to express one's reasoning. Another aim is that reasoning and expressing be joined dialectically. As students attempt to refine expressions of their reasoning, they have occasion to clarify their reasoning. As they clarify their reasoning and attempt to express it, they inform the systematicity of their notational schemes. It is hoped that under these circumstances students will find occasions to experience the negotiations that establish notational conventions, and only then would they be in a position to appreciate the natural, productive tension between creativity and conventionality. To understand a convention qua convention, one must understand that approaches other than the one adopted could be taken with equal validity. It is this understanding that separates convention from ritual.

To have students recreate conventionality wherever it might occur is neither practical nor desirable (Cobb, 1991). This would be like having them create their own languages to appreciate the conventionality of language. On the other hand we cannot ignore convention; to ignore convention in our teaching can lead students to think of mathematics ritualistically. We must choose judiciously those curricular sites where we address matters of convention honestly and directly. One principle to guide our choice is: address matters of convention in areas that are (a) conceptually central to the mathematics curriculum, (b) amenable to students' appreciation of convention within the area, and (c) propitious for students' recognition of convention generally. The areas of representing numerical value and representing methods of numerical evaluation are natural sites in the elementary curriculum fitting these criteria.

We cannot investigate the relative benefit of all possible treatments of convention in mathematics even within areas for which we decide convention might be treated productively. On the other hand, mathematics education research can attempt to make explicit those conventions that are assumed and treated as given, those conventions that are assumed and presented as conventions, and those conventions that are meant for students to recreate or to create in some idiosyncratic form.

THE STUDY

The study was designed to investigate what features of students' engagement in tasks involving base-ten blocks contribute to students' construction of meaning for decimal numeration and their construction of notational methods for determining the results of operations involving decimal numbers. The study was conducted over nine days—one day for pretest, seven days of instruction, and one day for posttest.

Conventions of Notation and Method

Base-ten numeration as a convention was not addressed directly. Rather, it was addressed thematically by encouraging students to refer to types of blocks by the number-name of their numerical value (e.g., hundred instead of flat). Base-ten numeration as a system for denoting numerical value was addressed as principle. It was not developed as one of many alternative numeration systems.

Nothing was taken as conventional about methods for solving arithmetic problems using base-ten blocks (e.g., start with blocks of smallest value). Students were free to approach problems of addition and subtraction without constraint, except that they were to solve the problems posed and they were to remain within the base-ten numeration system. Placement of initial terms in a notational statement of an addition or subtraction problem were explained as convention. The discussion is summarized in Figure 1. The purpose of having this discussion with students was to orient them toward a conceptual organization of the notational schemes they already possessed. Teachers used this convention initially, but they did not demand that students use it.

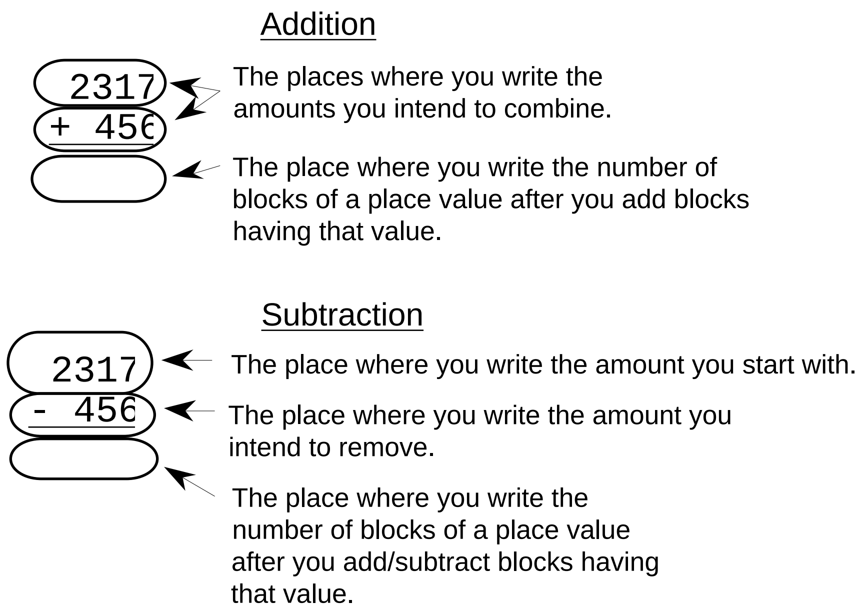


Figure 1: Conceptual organization of standard conventions as used by teachers and as explained to students.

Instruction was designed to give students as much freedom as possible to develop their own concrete methods for solving addition and subtraction problems with blocks, and to develop their own schemes for capturing their methods in notation. That is, instruction was designed with the intention that students reason naturally about solving problems with blocks-as-quantities and that they reflect their reasoning in notation. Care was taken to communicate to students that they were free to solve a problem in any way that made sense to them. Care also was taken to communicate to students that, when solving a problem, they had the freedom to organize their writing in any way they wished as long as a) whenever they took a concrete action with blocks they represented it in notation, b) whatever they wrote, at the moment written, expressed an action taken in their solution and the transformation in the blocks that the action caused, and c) their cumulative record expressed a summary of all actions taken up to that point in time.

Question

Suppose instruction emphasizes freedom of method for solving addition and subtraction problems with base-ten blocks and emphasizes freedom of notational expression within the constraint that expression accurately reflects meaning and method.

What aspects of students' interactions with base-ten blocks in the context of this instruction support their accomplishment of using notation to represent meaning and method?

Hypotheses

1) Fourth-grade students who have experienced typical mathematics instruction in their previous three years will not be easily convinced that they have the freedom to use notation creatively.

2) When students make a commitment to using notation to express their meaning and reasoning, settings that orient them continually to reciprocal relationships between what they have in mind and their expressions of it will be more effective in their development of meaningful use of notation than settings that do not continually orient them to these relationships.

MATERIALS

One way to provide a setting that continually orients students toward reciprocal relationships between what they have in mind and their expressions of it would be to have one mentor per student; the mentor would do nothing but ask, "What did you mean by that?" or "What do you have in mind?" (Heller & Hungate, 1985). Another way to provide such orientation is by designing concrete materials that provide constraints on students' concrete actions in places that are likely to draw their attention to relationships among meaning, notation, and expression. I chose the latter approach.

I created a mouse-driven, computerized microworld, called Blocks Microworld, that presents base-ten blocks and decimal numeration as linked systems (Figure 2). Any change in blocks causes a change in the numeral; any change in a numeral causes a change in blocks. The microworld was designed with one restriction not available in physical base-ten blocks. Students could perform representation-transforming actions on blocks (sometimes called trades) only by acting on digits in a quantity's numeral representation. Blocks microworld is an example of a mathematical microworld (Thompson, 1985, 1987) that employs multiple, linked notational systems (Kaput, 1986, in press; Thompson, 1989).

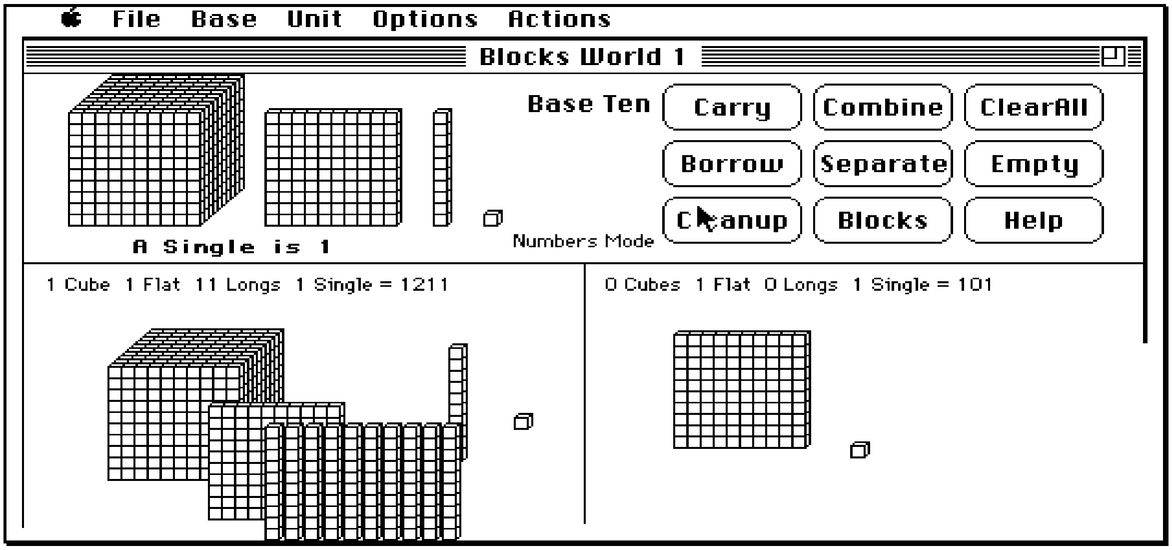


Figure 2: Screen display of Blocks microworld.

Students create collections of blocks by using a mouse to drag copies of blocks from the source region (at the top of the screen) to the region in which the blocks are to be stored. To trade one block for ten of the next smaller block, students clicked on the appropriate digit, which then became highlighted, and then clicked on the button labelled borrow. When students clicked borrow they saw one block of the appropriate type explode into ten of the next smaller type. To trade ten blocks for one of the next larger block, students clicked on the appropriate digit[1], which then became highlighted, and then clicked on carry. When students clicked carry they saw ten blocks of the appropriate type implode into one of the next larger type. Proposing to students that they use the ready-made actions associated with clicking borrow and carry was to propose a convention—one alternative being to get rid of what you have and replace it with what you want, as one would do with wooden base-ten blocks.

Blocks microworld was designed so that students could combine collections of blocks in a number of ways. One way would be to treat them the same as wooden blocks, dragging a collection in one region into the other region (one block at a time, several at a time, or all at once). The numeral representations of collections in the two regions update automatically. Another way to combine two collections is to put them into their separate regions and click combine. Upon clicking combine, the middle line disappears, the two collections become one, and there is one numeral display that corresponds to the combined collection.

The only natural way for students to separate one collection into two was to drag blocks from one region into another. They would create the minuend collection in one region and then drag an appropriate subtrahend subcollection to the other region. The numeral display updated automatically as they dragged blocks from one region to another.

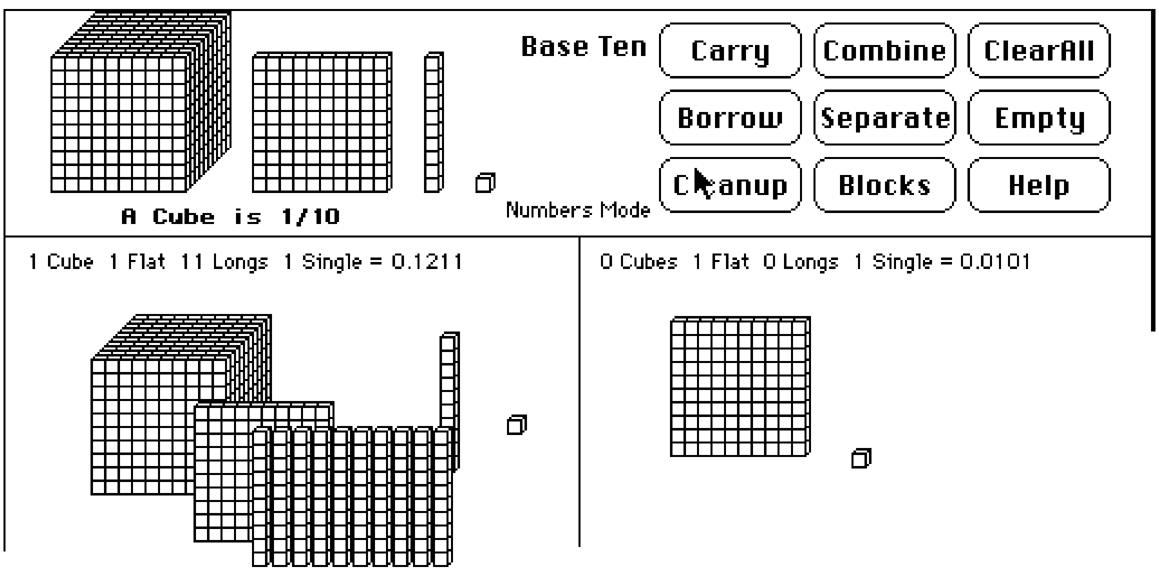


Figure 3: Screen display of Blocks microworld after a student has selected A cube is 1/10 in the Unit menu.

Blocks microworld proposes the full decimal numeration system by including a unit menu. The unit menu contains options for what block has a value of one. The options range in sequential powers of ten from “A single is 1000” to “A cube is $\frac{1}{1000}$”. The numeral display reflects the user’s choice of unit. Figure 3 repeats the display presented in Figure 2, except that a cube has a value of $\frac{1}{10}$, whereas in Figure 2 a single has a value of one.

The design of Blocks microworld is intended to support students' continual making of meaning for their notational actions and interpretation of notation. This support springs from their being oriented continually toward notation even when their intention is to manipulate blocks. Figure 4 illustrates the experience intended by the design of Blocks microworld versus the kind of experience that can be supported by actual base-ten blocks.

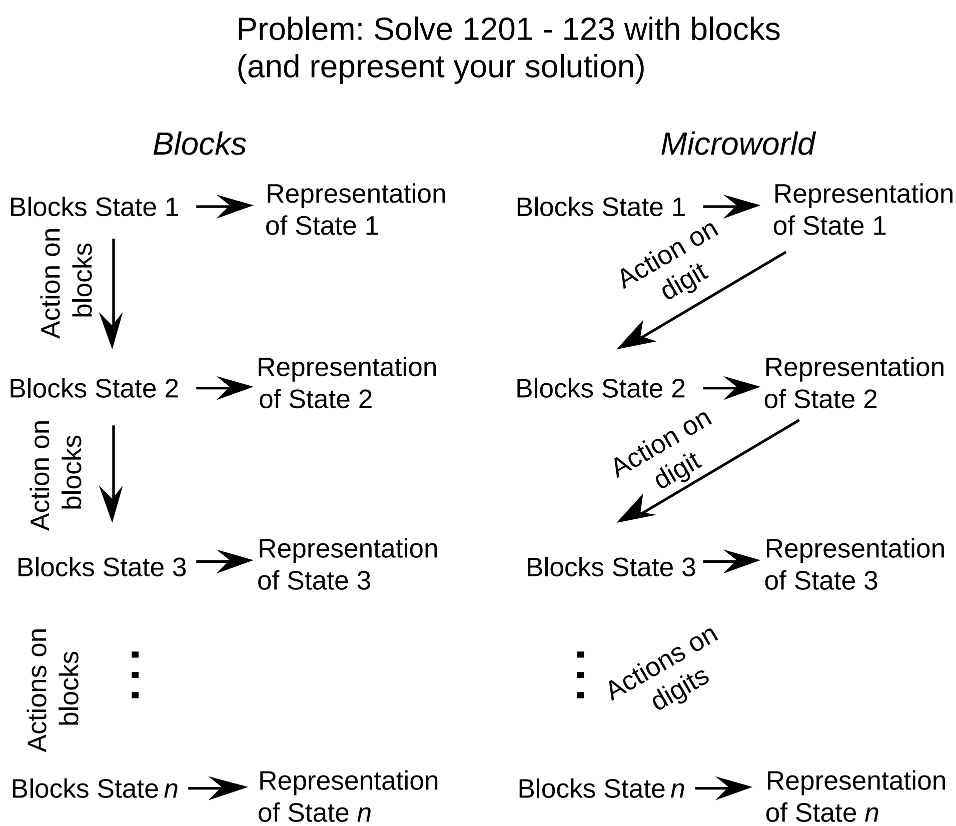


Figure 4: Differences in students' engagement with notation when working with wooden base-ten blocks and with Blocks microworld.

While solving a subtraction or addition problem with wooden blocks, a student's attention is oriented naturally toward the blocks themselves and his or her actions on them. However, the student's orientation does not entail a systematic, mechanical relationship between actions on blocks and notational actions—doing things to notation in anticipation of making something happen to a referent. Instead, the predominate use of notation when using wooden base-ten blocks is to represent a state of the blocks after the student somehow changes them. Changing the blocks themselves, in the students' experience, is unrelated to notation.

On the other hand, the design of Blocks microworld orients students to notational representations as things to be acted on in order to effect changes in blocks. The student intends to act on blocks, but that intention can be carried out only through actions on notation. It is hoped that, by this design, both blocks and numerals will be present in the student's experience at the moment of making a decision to act, and the student's decisions will be made according to systematic relationships between blocks, intended actions on blocks, numerals, and actions on numerals. One essential component of the students' understanding for which the program provides minimal orientation is the relationship among blocks, numeral, and numerical value. I designed the program with the assumption that this orientation will be supported by a teacher's instruction.

METHOD

Subjects

Twenty 4th-grade students enrolled in a midwestern university laboratory school were subjects of the study. Ten students were male; 10 were female. The laboratory school's enrollment is chosen to represent the geographic region's population academically and socio-economically. Average percentile ranks for subjects' Iowa Test of Basic Skills scores were: Concepts-70, Problem Solving-73, Computation-60, and Total Math-71. None of the students had studied decimal fractions, and none had used computers in studying mathematics.

Procedures

Students were assigned to two treatments: microworld instruction and wooden-block instruction. One group used Blocks microworld in instruction (microworld group), the other used wooden base-ten blocks (blocks group). The microworld group was taught by their regular fourth-grade teacher; the blocks group was taught by a research assistant.

Pretest and Assignment to Treatments

Students were matched according to their scores on a 19-item pretest covering whole number computation, place value, and fractions (test-retest Pearson $r = 0.82$). Item scores were entered into a stepwise multiple-regression analysis with total test score as the dependent variable. The analysis ended with six items being included in the regression equation. Sums were computed on those six items to give a pretest subscore; these six-item subscores were ranked in descending order. Pairs were formed by taking adjacently ranked sums. One member of each pair was assigned at random to the blocks group. The other member was assigned to the microworld group.

Two procedures were used to test the robustness of the rankings. First, students' six-item subscores were correlated with their total pretest scores (Pearson's $r = .92$). Second, item scores were analyzed by factor analysis. Two factors emerged: Representations and procedures. Factor scores were computed for each student. Students' factor scores were correlated with their six-item subscores (Pearson's $r = .91$).

Posttest

The posttest contained two parts: the pretest together with items on ordering decimals, decimal representations, appropriateness of method, and decimal computation. Items were scored for correctness of result and validity of method. Following the posttest, eight students were interviewed: two pairs scoring highest on the pretest and two pairs scoring lowest on the pretest. All interviews were videotaped and transcribed.

Instructors

The students' regular fourth-grade teacher taught the microworld group. A research assistant taught the wooden-blocks group. The regular fourth-grade teacher had never used this instructional approach before, nor had she used a microcomputer in instruction. Classroom observations of her instruction prior to the experiment suggested that her instruction was oriented to having students become skilled at performing prescribed procedures. Her instruction contained no orientation to conventions or meaning. The research assistant was an experienced teacher who was thoroughly familiar with the aims of instruction and with the computer program being used by the microworld group.

The two instructors and I met on four occasions prior to the experiment to plan instruction. These meetings were also used as a context in which to familiarize the microworld teacher with the computer program, its use during instruction, and to familiarize her with the aims of instruction.

Instruction

Instruction occurred over seven school days in May of 1988 during students' regular mathematics class. A posttest was administered on the eighth day. Each instructor worked from a script written for the microworld group. The wooden-blocks instructor modified segments that were appropriate only for the microworld so that they were appropriate for wooden blocks. The script was highly detailed—it directed what to demonstrate, what to say, and what to ask. I should note that at the same time that the script presented constraints on the teachers' actions, it was oriented toward generating contributions from students. For example, in the first lesson the teachers presented 4123 with blocks and wrote the standard numeral for it, asking students to explain the correspondence between blocks and numeral. The teachers then traded 10 flats for one cube and asked students two questions: "Has the total number of singles changed? What could we do to this number (numeral) so that it reflects what we started with, what we did, and what we ended up with?" After that discussion, the teacher then suggested an analogy: "We ask a farmer to deliver four thousand one hundred twenty-three apples. He brings three crates of one thousand apples, eleven crates of one hundred apples, two crates of ten apples, and three loose apples. Has he brought the correct number of apples? Does it matter how he grouped them?"

During student-centered activities both instructors interacted with students in typical ways. They circulated among the students—anwering questions, asking them to explain what they were doing, and responding to opportunities for remediation. These interactions were not controlled. Instructors remediated students' apparent difficulties in ways that were natural to the occasion in which they were noticed.

All microworld instruction was videotaped; field notes were taken during wooden-blocks instruction. Wooden-blocks instruction could not be videotaped because two special education students were present during instruction, but not part of the study. State and university policy did not allow videotaping of special education students without parents' permission.

Instruction followed the outline shown in Table 1. The purpose of beginning with addition and subtraction of whole numbers was to provide a familiar context within which the distinction between representation-transforming actions and quantity-transforming actions could be operationalized, and within which issues of notation and use of notation to represent method could be raised. While it appears that decimal numeration was introduced in one day (Day 4), it was actually built into whole-number instruction thematically. Whole-number instruction on numeration made frequent reference to the reciprocal relationships ten of and one-tenth of between successively larger blocks.

In-class instruction was comprised of teacher-centered and student-centered activities. The microworld teacher used a Macintosh connected to a large-screen projector during class discussions. The blocks teacher used an overhead projector and plastic blocks during class discussions. Students in the microworld group used computers for in-class activities, with two students per computer; students in the blocks group used wooden blocks for in-class activities, with two students per station. Microworld students had not used a Macintosh before; it took just a few minutes for them to become skilled at using the microworld. All students had seen pictures of base-ten blocks in their textbooks, but had never used physical blocks.

Table 14
Outline of Instruction for Both Groups

Day	In Class	Homework
1	- Demonstration of microworld, Introduction to base-ten blocks - Representing quantities - Borrowing and representations of having borrowed - Action-meanings of subtraction - Representations of borrowing and subtracting	- Represent steps in various to subtraction problems; solutions presented as sequence of pictures of block collections
2	- Representations of borrowing and subtracting (Activity)	- Represent steps in various solutions to subtraction problems; solutions presented as sequence of pictures of block collections
3	- Carrying and representations of having carried - Action-meanings of addition - Representations of carrying and adding (Activity)	- Represent steps in various solutions to addition problems; solutions presented as sequence of pictures of block collections
4	- Units other than a single - "Ten of" and "one-tenth of" relationships among blocks and place values - Base-ten decimal notational system - Representing numbers in decimal notation (Activity)	- Represent given decimal fractions in three ways using pictures of blocks
5	- Subtraction of decimally-represented fractions - Representations of borrowing and subtracting	- Represent steps in various solutions to subtraction problems; solutions presented as sequence of pictures of block collections
6	- Addition of decimally-represented fractions - Representations of carrying and borrowing	- Represent steps in various solutions to addition problems; solutions presented as sequences of pictures of block collections
7	- Mixed practice on addtion and subtraction	- None
8	Testing	

Instruction on whole number addition and subtraction emphasized place-value numeration, transformations of numerals, the creation of methods for solving addition and subtraction problems, and the recording of actions done while applying a method. Students' freedom to create methods for operating on blocks to solve addition and subtraction problems was continually emphasized, with the provision that students had to represent in notation each and every action in their method, whether it was a change of a quantity's representation or a change in a quantity's value. How students denoted their actions was in large part left to them. Their personal methods for solving problems were discussed frequently, as were their their various notational schemes for representing a given method.

Homework for both groups focused on representing the concrete actions taken by someone who used non-standard methods to solve addition or subtraction problems. The methods were presented in sequences of snapshots of block collections as the mythical solver used the blocks to solve a problem. Homework problems started with a stem like this: “Jim solved 13.25 - 6.375 using blocks. What he did with the blocks is shown below. Use [paper and pencil] to record the steps in Jim’s solution.”

Analyses

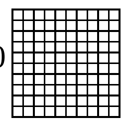
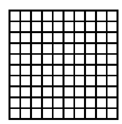
Posttest results were analyzed in two parts. The first part of the analysis examined change in students’ accuracy from pretest to posttest. The second part of the analysis examined students’ responses to questions dealing with content that was introduced in the experiment—decimal numeration and calculation with decimal numerals. Both analyses were done from two perspectives: performance and method. Analysis of performance focuses on the correctness of students’ answers. Analysis of method focuses on whether and in what ways students’ use of notational methods were influenced by instruction. Finally, a subset of students were interviewed in order to assess students’ abilities to relate their written work to some mathematical principle or concrete model.

RESULTS

Change in Accuracy from Pretest to Posttest

Table 2 shows the relationship between students’ pretest and posttest. on whole-number computation items, fractions, and ratio nature of decimal fraction. There was no substantial improvement on any items. There was slight improvement in students’ understanding of the ratio nature of decimal fractions, as shown by the last three items of Table 2.

Table 2
Number of Correct Responses on Pretest and Posttest Items
for Blocks and Micrworld Groups

	Blocks		Microworld	
Item	Pre	Post	Pre	Post
3004 - 286	5	6	5	5
7814 + 2648	9	10	9	9
5002 - 493	5	5	4	5
Shade this grid so that $\frac{2}{10}$ of the grid is shaded. 	6	6	7	7
Shade this grid so that $\frac{3}{5}$ of the grid is shaded. 	3	2	2	2
If a flat stands for one, then what does a long stand for?	3	5	2	5
If a flat stands for one, then what does a cube stand for?	5	5	3	5
If a flat stands for one, then what does a single stand for?	3	5	3	5

Note. n = 10 for each group.

Decimal Computation and Decimal Numeration

On the remainder of the posttest the blocks students were generally more accurate on decimal computation items than were microworld students, but they were less successful than microworld students on two of three conceptual items.

Table 3 shows students' performance on decimal computation items. I could not interpret the blocks groups' three errors on the first three tasks. The microworld students' errors on the first three items were: Minor errors such as "2+5=8" (5 errors), what might be called "decimal point separates whole numbers" errors, such as "7.31 - 6.4 = 1.27" (3 errors), and 3 errors that I could not classify.

Table 3
Number of Correct Responses for Decimal Computation Items on the Posttest for Blocks and Microworld Groups

Item	Blocks	Microworld
$\begin{array}{r} 12.27 \\ \underline{+\ 5.84} \end{array}$	10	6
7.31 - 6.4 = ______	8	5
$\begin{array}{r} 8.03 \\ \underline{-\ 2.9\ } \end{array}$	9	8
14.8 + 7.23 = _____	4	2

Note. n = 10 for each group.

Six errors were made by blocks students on "14.8 + 7.23 = __." These were minor errors (2), alignment of numerals on their right hand side (3), and a mixture of right-hand alignment and treating the decimal point as a separator between two whole numbers (1). Microworld students made eight errors on this item. They were minor errors (3), no answer (1), treating the decimal as a separator between two whole numbers (3), and adding as if the two addends were whole numbers (1).

Much of the instruction on decimal numeration emphasized converting tens into ones, ones into tenths, and so on, in the process of changing a quantity's representation while leaving its value the same. Two items in Table 4 were intended to assess students' abilities to determine equivalence of representation. Microworld students were more successful on the first item than were blocks students, while blocks students were more successful on the second item. Students' scratch work, and in some instances the lack thereof, indicates a reason for this discrepancy. None of the blocks students wrote any scratch work for either part of this item. Five microworld students wrote scratch work for the first part (e.g., $\frac{32}{100} = \frac{3\cancel{0}}{10\cancel{0}} + \frac{2}{100}$). Three microworld students wrote scratch work for the second part—all making a mistake in converting from tenths to hundredths to thousandths. Thus, it appears that at least five microworld students attempted a numerical conversion of fractions into decimals. Blocks students may have looked for a visual match between “4, 3, 2, 5” and the presented fractions.

Table 4
Number of Specific Responses to Decmial Fraction Equivalence and Order Items on the Posttest

Stem		Response	Blocks	Microworld
4.325	$4 + \frac{32}{100} + \frac{5}{1000}$	Same	3	7
		Different	4	2
		Can’t Tell	3	1
4.325	$4\frac{325}{1000}$	Same	6	4
		Different	3	4
		Can’t Tell	1	2
7.89 is smaller than 7.9		Yes	3	6
		No	6	3
		Don’t Know	1	1

Note. n = 10 for each group.

Table 4 also contains one item given to assess students’ understanding of numerical ordering of numbers represent as decimals. There is a notable difference between blocks and microworld performance. None of the students provided scratch work to suggest their reasoning (see Individual Interviews).

Solution Methods

A large amount of instructional time was spent having students discuss methods for representing actions they took while solving addition and subtraction problems in concrete settings. I examined students’ written work on calculation items to determine what influence these discussions and activities might have had on their notational methods. Students’ notational methods expressed in their solution to pretest whole-number calculation items served as a baseline. Students’ notational methods on those same items were examined on the posttest.

Whole-number subtraction methods expressed on the pretest fell into two groups: standard subtraction and buggy subtraction (see Figure 5). A method was classified as standard subtraction whenever 0 in the minuend numeral was replaced immediately with 9. Buggy subtraction is an incorrect modification of standard subtraction.[2]

Whole-number subtraction methods on the posttest fell into three groups: standard subtraction and buggy subtraction, as before, and novel subtraction. Novel subtraction included two methods: elaborated subtraction and expanded subtraction (Figure 5). Elaborated subtraction, while appearing as perhaps a variation on standard subtraction, is considered novel because of the evident attention it shows to changes of representation while holding quantity invariant. Expanded subtraction is truly novel relative to students' methods on the pretest. The student employing this method evidently attempted to represent the removal of 286 from 3004 where each was represented in expanded notation, and attempted to reflect the operands of representation- and quantity-transforming actions. For instance, in the method shown in Figure 5, after taking away 200 from 1000 in the hundreds line, leaving 800, the student then indicated a borrowing action upon 800, which indeed represented the number of hundreds remaining after taking away 2 hundreds, and then indicated a corresponding change in the tens part of the minuend.

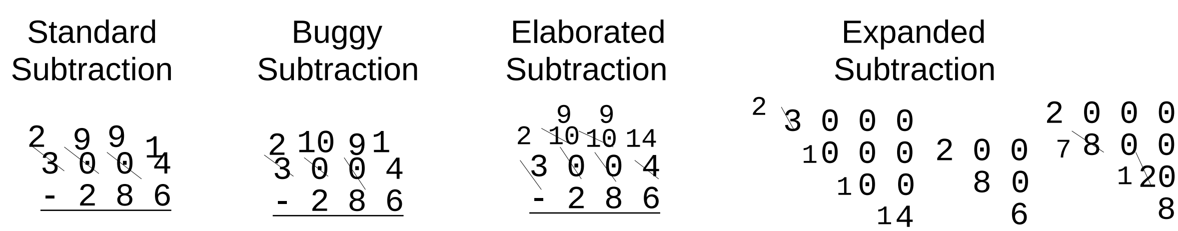


Figure 5: Pretest and Posttest subtraction methods.

Each blocks student expressed the same whole-number subtraction notational method on the posttest as on the pretest. Eight of 10 microworld students used the same methods on the posttest as on the pretest. Two microworld students changed from standard subtraction or buggy subtraction to a novel method. Results for 5002-483=___ were the same.

Standard addition was the only notational method for addition expressed by students on the pretest (Figure 6). Two methods were expressed on this same item on the posttest: standard addition and add within columns (Figure 6). Add within columns is a method by which one records the addition of two quantities' place-value constituents. One records the total number of ones, the total number of tens, etc. and then conventionalizes the resulting numeral. This was a commonly-used method among students during instruction. Every student used standard addition on the pretest. Seven blocks students retained their use of standard addition on the posttest; three changed to add within columns. Six microworld students retained their use of standard addition on the posttest; four changed to add within columns.

```
Standard Addition      Add Within Columns
  1   1
  7 8 1 4                7 8 1 4
+ 2 6 4 8              + 2 6 4 8
---------              ---------
1 0 4 6 2                9 1 4 5 1 2
                       1 0 4 6 2
```

Figure 6: Two methods of whole-number addition used by students on the posttest.

Of the four decimal calculation items (see Table 3), only the two items on addition showed any difference between groups. I did not have the foresight to include decimal subtraction problems that would have made different notational methods evident, such as 10.03 - 6.004 = __. I will use the labels standard addition and add within columns as applied to whole-number numerals (Figure 6) to name their natural extensions to decimal numerals. Figure 7, which replicates one microworld student's written work, is an example of add within columns applied to decimal numerals. This student first added within each place value, getting 1 ten, 11 ones, 10 tenths, and 3 hundredths. She then changed 10 tenths into 1 one, producing 12 ones. Her last action was to change 1 ten and 12 ones into 2 tens and 2 ones. Her answer can be read from the lowest digit in each place-value column.

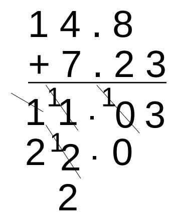


Figure 7: Add within columns addition method for decimal addition.

Table 5 shows a count of the various methods used by students on decimal addition items. Align right means that a student aligned numerals on the right-hand side. Blocks students were more likely than microworld students to use standard addition or align right on decimal addition problems, whereas microworld students were more likely to use add within columns. Also, two of the three microworld students who showed no work gave answers that were suggestive of an add within columns method.

Table 5. Decimal Addition Notational Methods on Posttest.

Item	Method	Blocks	Microworld
12.27 + 5.84	Standard Addition	7	2
	Add within columns	3	8
14.8 + 7.23 = __	Standard Addition	5	0
	Add within columns	2	5
	Align right	3	1
	Novel	0	1
	No Work	0	3

Note. n = 10 for each group.

Results on whole-number instruction suggest students' resistance to modifying their already-automatized notational schemes. One indicator of relative impact of treatments on that resistance is students' willingness to abandon their automatized schemes in the context of new material. Seven blocks students retained whole-number standard addition as their method for decimal addition. Three blocks students used add within columns for both whole-number and decimal addition. Four of six microworld students who used standard addition for whole-number addition changed to add within columns for decimal addition. Two microworld students used standard addition for both. Four microworld students used add within columns for both. Given that standard addition was not taught (indeed, it rarely came up in discussions), it seems safe to presume that use of standard addition on decimal addition suggests assimilation of decimal addition to whole-number notational schemes. Blocks students were more likely to have assimilated decimal addition to whole-number notational schemes.

Finally, one item was given to assess the extent to which students felt instruction on alternative methods conflicted with prior conceptions of what constitutes school mathematics. The text of this item is shown in Table 6, along with students' responses. Blocks students tended to feel that standard addition was the right way to do addition, even if other ways give the same answer. Only one microworld student agreed with this statement. On the other hand, most microworld students disagreed with the statement, whereas only two blocks students disagreed with it.

Table 6
Children's Attitudes about Correctness of Unprescribed Methods.

Statement	Response	Blocks	Microworld
This is the RIGHT way to add 8276 and 4185. Other ways might give the same answer, but they are not the right way: $\begin{array}{r} {}^{1\,1} \\ 8276 \\ +\ 4185 \\ \hline 12461 \end{array}$	Yes	5	1
	No	2	7
	Don't Know	3	2

Note. n = 10 for both groups.

Interviews

Four students scoring highest on the pretest and four students scoring lowest on the pretest were interviewed after instruction and posttest. The interviews focused on students reasoning as expressed on the posttest.

Low, Blocks Group

Students in this group gave explanations that were largely procedural, without reference to actions on blocks or to constraints imposed by the decimal numeration system. The following excerpt illustrates the sorts of explanations they gave.

I: I was just wondering why you didn't do it this way ? Why you did it that way … Why did you line up the seven and the six ...

$$\begin{array}{r} 7.31 \\ \underline{-\ 6.4} \end{array}$$

T: Well, I guess because, because you can't, well cause you need to line up the decimals, because you can't subtract. Well, its four minus, four minus one and decimal. And three minus six. But there's not another decimal, so you can't even do that. The decimals have to be lined up.

I: You mean the decimal points?

T: Yeah.

I: Ah. And you don't think it's correct if I say ,"Okay, one minus four, I can't do that so I can borrow from here, and then make this an eleven. Eleven minus four is seven and then let's see. Twelve minus six is six. Then put a decimal here. And then say, six." That wouldn't be right?

T: Well, no, because you'd still have this decimal hanging there, and you wouldn't, you couldn't put that there because it wouldn't, then it would be seven point six point … six point seven [i.e., 7.6.7].

Low, Microworld Group

Two students in the Low, microworld group missed two of the three days of instruction involving decimal numeration—the day on which numeration of decimal fractions was introduced and the following day on subtraction. This certainly contributed to their low posttest scores. Students in this group showed little understanding of decimal numeration, and had lost even the modest facility they previously had with written computation.

I: Which one (7.9 or 7.89) is bigger than which?

B: This one's bigger than that one [*pointing to 7.89*].

I: Seven point eight nine is bigger than seven point nine?

B: Yeah

I: Why?

B: Because you have an eight nine and you just have a nine there.

I: And?

B: And eight nine is bigger than nine.

I: Eight nine is bigger than nine. What place value is this? Where is this eight?

B: Tenths.

I: In the tenths? And this one here?

B: Is nine ones.

High, Blocks Group

There was no evidence that either of the two High, blocks students had formed connections among decimal numeration, base-ten blocks, and notational conventions. These students evidently modified their whole-number procedures for processing numerals to accommodate the presence of a decimal point. The reader should be aware that the student quoted in this excerpt received the second highest total posttest score.

I: Remember the block stuff? What did you think of it?

J: Um, I thought it was pretty easy because we had done it in our math books before.

I: You did it in your math book? Did you do decimals in your math book?

J: No.

I: Did the blocks help you think about decimals and fractions with tenths and hundredths?

J: No, not really.

I: Not really, huh. So how was it that you learned about decimals? You know, to add and subtract them.

J: I don't know.

I: You don't know. What were you thinking? Did it make sense to you?

J: Well, I just thought about it as doing it without the decimal and then I just added the decimal at the end.

I: …So you just ignored the decimal and then treated it...

J: Yeah.

I: ...like you always did before.

J: Sort of like dollars and cents.

I: Ah, I see.

J: So you don't have to worry about it.

(Portion of excerpt omitted.)

I: *(Discussing "7.31 - 6.4 = ____.")* Okay. Down here you wrote, let's see … You wrote them up and down and then you crossed out the three and wrote thirteen. Could you tell me why?

J: Because I took one hundred away and then put it in the tens.

I: Are you putting a "d" or a "th" at the end? Are you saying hundred or hundredth?

J: Hundred.

I: With just a "d" at the end?

J: Yeah.

(Portion of excerpt omitted.)

I: Here you said "yes", that seven point eight nine is smaller than seven point nine.

J: Because, um, you would just have to put a zero here. So it would be seven point nine, zero.

I: Seven point nine, zero. Why would you put a zero there?

J: Um.

I: How would you read this number in hundreds, tens, ones, tenths, hundredths, stuff like that?

J: Um, seven hundreds, eight tens and nine ones.

I: That's seven point eight nine. It's seven hundreds, eight tens, and nine ones. Right?

J: Yeah.

I: Okay. And seven point nine?

J: Um, seven hundreds, nine tens, and zero one.

I: So this is the way to write seven hundred and ninety … seven nine with no zero at the end.

J: No.

I: No? But you told me that that is seven hundreds and nine tens...Is that right?

J: No.

I: No? Which one of these is smaller?

J: This one (7.9).

I: The seven point nine. What does that seven point nine stand for, what number?

J: Seven tens and nine ones.

High, Microworld Group

The two High, microworld students showed facility with decimal numeration and appeared to have made sense of sources of conventions. One of them made a relatively large number of errors due to unorganized, messy writing. The other High, microworld student, quoted in the following excerpt, received the highest total posttest score.

I: Some people have said that they think that this problem should be written like this. So when you write it up and down, seven point three one take away six point four, it looks like this. They say you always have to line up the right hand side.

$$\begin{array}{r} 7\,.\,3\,1 \\ \underline{-\ 6.4} \end{array}$$

K: Well, not exactly. If you have a decimal point on this side, you got to…um…you got to match the decimal point with the…this…and this is the ones and this is the tens. So you're actually measuring up by the decimal point.

I: Well, it sounds like we have two rules that are in conflict. One is to always write…line things up on the right hand side, and the other is to always line up the decimal point. Are those different rules?

K: Yeah. Um…when you line them on up at the right hand side, you usually have two whole numbers which are ones and other…other numbers in front of it. And so when you're adding them or subtracting to the other, they all fit together. But…when you have a decimal point and there's this number…has a, a one, a number that is a hundredth and this one is only a tenth, you scoot it over to match…the decimal point.

I: Oh, I see. So lining up the decimal point is what keeps you from...

K: Messing up.

(Portion of interview omitted.)

I: Did you feel like you knew what you were doing?

K: Um-hum (yes).

I: When you were doing these problems, did you ever think about the, the blocks?

K: Yeah, sometimes. When I got stumped or stopped.

Classroom Observations

Students in the microworld group responded differently to instruction than did students in the blocks group in regard to their openness to new methods and perspectives.

While the posttest showed little influence on microworld students' preferred methods for whole-number operations, analyses of videotaped lessons show students using quite creative methods during instruction. For example, one student decided to borrow from every digit in a minuend, subtracted within columns, and then carried within the difference if any digit was greater than 9. Evidently, microworld students did not have confidence in their new methods at the time they took the posttest.

We found nonstandard uses of notation in blocks students' written work, but only in those cases where they were directed to represent in notation someone else's solution. My observations during blocks instruction lead me to believe that students were missing the point of the instruction. I do not interpret this as a natural outcome of using base-ten blocks. Rather, I see it as a natural outcome of giving them the freedom to create their own notational methods when at the same time their experience of constraints on their use of notation was infrequent (i.e., when their instructor looked at their work).

On several occasions it was evident that wooden blocks were of limited value in constraining students' actions and reasoning relative to correspondences among assumptions, meaning, and notational actions. One in-class activity directed students to "Select a block to stand for one, then put blocks out to represent 3.41." One blocks student selected a cube to stand for one, then looked back at her paper, reading "three hundred forty-one." She put out three flats, four longs, and one single, looked at the paper and back at the blocks, then went on to the next task. A student in the microworld group started similarly, selecting <u>A Cube is One</u> from the <u>Unit</u> menu, and then put out three flats to make "three hundred forty-one." After putting out three flats, the microworld student looked at the screen and said, "Point three? That's not what I want. ... Oh! A cube is one!" Blocks microworld's design allowed students to experience occasions where they realized the necessity to adhere to their assumptions and to constrain their reasoning according to their assumptions.

Though the wooden-blocks teacher frequently oriented students toward correspondences between what they did with blocks and what they might write on paper, the blocks students showed little evidence of feeling constrained to write something that actually represented what they did with blocks. Instead, they appeared to look at the two (actions on blocks and writing on paper) as separate activities, related only tangentially by the fact that the written symbols could have reference in the world of wooden blocks.

Students in the microworld group repeatedly made references to actions on symbols as referring to actions on blocks (e.g., “Borrow a thousand so that we can break up a cube”). One reason for this might be that their attention was always oriented toward the symbols, even though their intention was to operate on blocks. That is, the microworld students acted on blocks by acting on their notational counterparts. Thus, relationships between notation and materials appeared to be prominent in their experience.

A severe problem encountered in the study was that students plainly disliked having the freedom to create their own methods. It was difficult obtaining their commitment to reason, which was the only basis from which instruction could begin conceptually. Students in the blocks group showed resistance throughout instruction to entertaining alternative methods of solving addition and subtraction problems. On one occasion a student asked, “Can we just do it the old way?” The microworld group resisted discussing alternative methods, but resistance evidently diminished over the seven days of instruction. I cannot attribute attribute the change in the microworld group to the instructor, for she herself had to be reminded during the teaching experiment that creative methods were all right no matter how nonstandard they might be. One microworld student, when asked after class one day about what he thought about this unit, replied: “I really like it, this way of doing it if it makes sense to you. But I’m afraid to do it, really.” His fear was that “Next year the teacher might mark it wrong.” When his regular classroom teacher assured him this would not happen, he asked, “Would you have marked it wrong at the beginning of the year?” His teacher replied, “Yes, I would have.”

Finally, both groups quickly adopted similar notational methods for whole-number addition. The metaphor of combine the quantities evidently provided a simple enough image that they could accomodate their existing notational schemes to it. In the case of subtraction, there was considerable variability in microworld students' methods; in fact there was considerable variability within individuals from problem to problem. No consensus emerged among microworld students as to preferred methods, either for enacting subtraction with blocks or for notational schemes for their methods. Students in the blocks group used standard subtraction whenever they had the freedom to use their own method in absence of blocks. It was also common to see students using standard subtraction as their notational scheme when representing their actions on blocks, even when standard subtraction did not model their actions with blocks.

DISCUSSION

The lack of substantial impact of either treatment on students' already-automatized whole-number subtraction methods or on their accuracy on whole-number addition and subtraction items is consistent with past studies—both those attempting to teach prescribed notational procedures (Resnick & Omanson, 1987) as well as those attempting to have students construct their own notational methods (Labinowicz, 1985; Wearne & Hiebert, 1988). On the other hand, both treatments influenced a substantial minority of the students in regard to their already-automatized notational methods for whole-number addition, but in the case of the blocks group the effect appears to have been entirely syntactic.

The differential impact of instruction on subtraction and addition methods might be due to the relative simplicity, both syntactic and conceptual, of the add-within-columns method. On the other hand, subtraction methods are complex both notationally and conceptually. A complex procedure that has been memorized as a ritual is harder to modify than a simple procedure that has been memorized as a ritual.

If students memorize a procedure meaninglessly, it is extremely difficult to get them to change it, even with extended, meaningful remediation. Also, it should be noted that those students who did change their notational methods to express their reasoning tended to be less accurate in their calculations than those students who had already automatized a correct method and continued to use it. Newly-constructed methods are unstable. The students who used novel methods had relatively little time to make them automatic.

Students in the blocks group evidently assimilated instruction on decimal numeration and operations with decimal numbers into structures they had already developed for whole-number numeration and operations on whole numbers. This is consistent with the findings of several studies (Balacheff, 1990; Nesher & Peled, 1986; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989), but it is inconsistent with the findings of Wearne and Hiebert (1988). Wearne and Hiebert found that instruction similar in spirit to that used in this study influenced fourth-, fifth-, and sixth-grade students' intention and ability to analyze situations in terms of meanings for symbols used in the presentation of tasks. One source of discrepancy between the results of this study with the blocks students and those of Wearne and Hiebert's study could reside in the different lengths of instruction. Students in this study spent three lessons on decimal numeration and operations on decimal numbers; students in Wearne's and Hiebert's study spent nine lessons on these topics. Also, instruction in this study was structured with the intent that students generalize their numeration and operation schemes from whole numbers to decimal numbers. Instruction in Wearne's and Hiebert's study began with decimal numeration. What instructional use they made of relationships between whole-number numeration and decimal numeration is not clear from their report. Also, it is not clear from their report how much attention was given to notational methods in general, and it is not clear what attention was given to the problems requiring complex notational actions. Their report did not contain addition items that involved regrouping, and it did not contain any subtraction items.

There were differences between blocks and microworld groups in calculation methods with decimal numbers, concepts of decimal numeration, and in their attitudes toward uses of notation. Microworld students were more likely to use methods suggestive of meaningful use of notation, and their answers to questions having to do with decimal ordering and decimal equivalence also suggested that their answers emanated from trying to understand the questions by principles of decimal numeration. They were often inaccurate, but many of their attempts appeared to be based on principles of numeration.

Microworld students' inaccuracy on decimal calculation items is consistent with the effects of being in a state of disequilibrium, where they are making sense of new material but have not fully consolidated their sense-making activities into stable cognitive structures. Klahr and Siegler (1978) reported a similar finding, wherein students gaining sophistication with the concept of balance were more inaccurate in their predictions than younger, less sophisticated children. The older children understood more about the task, and that made the task more complex for them. It takes longer to construct a rule from the basis of understanding than it takes to memorize a rule that simplifies what might otherwise be a complex conceptual task (e.g., line up the decimal points and proceed as with whole numbers).

Microworld students' state of disequilibrium can be taken as positive evidence that their instructional experiences caused them to grapple with nontrivial ideas. It may seem odd to consider computational disequilibria as a positive outcome. However, given that students' initial use of notation is often ritualistic and impervious to remediation, and considering the sophistication of the notational issues we asked them to address explicitly, we should expect that they will be confused when we change the contract between them and the subject in deep and fundamental ways. What we now need to understand is how such disequilibria might be resolved into stable, principled schemes.

Although microworld students were more likely to make novel use of notation, neither blocks students nor microworld students had constructed decimal numeration as a numerical system. However, microworld students appeared to be grappling with these ideas to a greater extent than were blocks students. Microworld students may have been more likely to envision actions on blocks as they worked within notation, but evidence from several tasks indicates that many students had not fully internalized the blocks themselves as representative of numerical value. This would suggest that much greater attention should have been paid to numerical relationships within base-ten numeration. Further studies need to begin with a much clearer foundation in whole-number numeration and the construction of numerical value. Also, the study was too short. On the other hand, it was too short for both groups. The study being too short cannot explain the differences in attitudes and understanding that were observed.

We must take care not to conclude from this study that uses of wooden base-ten blocks, or physical materials in general, are ineffectual in producing understanding of notational methods. There are too many studies that suggest otherwise (e.g., Fuson, 1990; Fuson & Briars, 1990; Sowell, 1989). Rather, this study would suggest that we reexamine previous studies, asking to what degree students have internalized procedures as prescriptions, as distinct from having internalized them as conventions.

There are two lessons suggested by this analysis and the results of the study. The first is that before students can make productive use of concrete materials, they must first be committed to making sense of their activities and be committed to expressing their sense in meaningful ways. The second is that for concrete embodiments of a mathematical concept to be used effectively in relation to a learning some notational method, students must come to see each as a reflection of the other—constraints and all. They must end up feeling just as constrained in their notational actions as they do with those actions' counterparts in a concrete setting.

FOOTNOTES

[1] I am using digit to refer to the denotation of how many of a particular position-value are in a quantity's representation. With this usage 12 can be a digit.

[2] Buggy subtaction might be described as a mixture of representation-transforming actions and actions emanating from blind application of the rule cross out 0 and write 9. However, characterizing it this way would impute "representation-transformation" as a mental operation to students doing buggy subtraction. This would be inappropriate, since its occurrence was common on the pretest.

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Patrick W. Thompson, Associate Professor of Mathematics, Center for Research in Mathematics and Science Education, San Diego State University, 6475 Alvarado Rd. #206, San Diego, CA 92120.