CONCEPTUAL ANALYSIS OF MATHEMATICAL IDEAS: SOME SPADEWORK AT THE FOUNDATION OF MATHEMATICS EDUCATION[1]

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Running Head: Conceptual analysis

Mathematics during the late 18th century through the early 20th century experienced a period of turmoil and renewal that was rooted in a variety of attempts to put mathematics on solid conceptual footing. Taken-for-granted meanings of concept after concept, from number to function to system, came under increasing scrutiny because they could not carry the weight of new ways of thinking. In a very real sense, that period of time can be characterized as mathematicians' search for broad, encompassing coherence among foundational mathematical meanings. Part of the resolution of this quest was the realization that meanings can be designed. We can decide what an idea will mean according to how well it coheres with other meanings to which we have also committed, and we can adjust meanings systematically to produce the desired coherence. Mathematics education is in the early stages of a similar period. Competing curricula and standards can be seen as expressions of competing systems of meanings--but the meanings themselves remain tacit and therefore competing systems of meanings cannot be compared objectively. I propose a method by which mathematics educators can make tacit meanings explicit and thereby address problems of instruction and curricula in a new light.

My apologies to non-U.S. readers of this article. What I say here is focused very much on problems that exist in the United States. My only excuse is that the problems are so great in U.S. mathematics education that vetting some of them publicly might provide useful insights for others to avoid similar problems elsewhere.

With that said, I start with three observations. The first is that students' mathematical learning is the reason our profession exists. Everything we do as mathematics educators is, directly or indirectly, to improve the learning attained by anyone who studies mathematics. Our efforts to improve curricula and instruction, our efforts to improve teacher education, our efforts to improve in-service professional development are all done with the aim that students learn a mathematics worth knowing, learn it well, and experience value in what they learn. So, in the final analysis, the value of our contributions derives from how they feed into a system for improving and sustaining students' high quality mathematical learning.

The second observation is that, in the United States, the vast majority of school students rarely experience a significant mathematical idea and certainly rarely experience reasoning with ideas (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999; Stigler & Hiebert, 1999). Their experience of mathematics is of procedure after procedure. This is not to say that various curricula do not aim for students to learn ideas. Rather, students do not experience any that are significant. By "significant" I mean ideas that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students' reasoning. Base ten numeration is a significant mathematical idea. If students have learned it well then, without being taught a procedure to do so, they can reason their way to answering the question, "How many hundreds are in 35821?" They can reason that there are 358 hundreds in 35821 because 5 thousands contain 50 hundreds, and 3 ten thousands contain 30 thousands, and therefore contain 300 hundreds. Or, they can reason that there are 358.21 hundreds in 35281 because 2 tens make two tenths of one hundred, and 1 one makes one hundredth of one hundred. This type of reasoning is rare in U. S. schools because students are not expected to develop mathematical meanings and they are not expected to use meanings in their thinking.

The third observation is that too many mathematics teachers at all levels spend too little time at the outset of teaching a topic on having students become steeped in ideas and meanings that are foundational to it. As Deborah Ball said at a recent conference, mathematicians and

mathematics teachers are too eager to condense rich reasoning into translucent symbolism. They are too eager to get on to the "meat" of the topic, namely methods for answering particular types of questions.

COHERENCE AND MEANING

The issue of coherence is always present in any discussion of ideas. Ideas entail meanings, meanings overlap, and incoherence in meanings quickly reveals itself. Thus, my talk will be about issues of coherence, incoherence, and meaning as much as it will be about mathematical ideas and analyses of them.

The word "coherence" and its derivates occurs with increasing regularity in mathematics education publications. The 1989 *Curriculum and Evaluation Standards* (NCTM, 1989) raised the issue only 8 times in 364 pages. The 2000 *Principles and Standards* (NCTM, 2000) raised issues of coherence 39 times in 402 pages. The 2006 *Curriculum Focal Points* raised it 16 times in 40 pages. The final report issued by the *National Mathematics Advisory Panel* (2008) raised it 19 times in 91 pages. One would think that with the increasing emphasis on curricular coherence, everyone would be clear on how to think about it. This is, unfortunately, not the case. Of all these documents, only the NMAP final report defines coherence, and only in regard to curricula.

> By the term *coherent*, the Panel means that the curriculum is marked by effective, logical progressions from earlier, less sophisticated topics into later, more sophisticated ones. (National Mathematics Advisory Panel, 2008, p. xvii)

I must confess my disappointment in the Math Panel's definition of curricular coherence. Effectiveness might be a consequence of coherence, but it cannot define it. Nor does coherence imply logical progression of topics, at least from a mathematical point of view (Thompson, 1995). Despite these, my disappointment might have been less had their definition been followed by examples of what the panel accepts as coherent curricula. But there were no examples. I suspect that even with examples my disappointment would have been the same, because the panel defined curricular in terms of topics, not in terms of ideas. Ultimately, coherence of a curriculum (intended, implemented, or experienced) depends upon the fit of meanings developed in it. Schmidt's (2002) example of Hong Kong's curriculum displays this very characteristic—the example, drawn from the topic of ratio, rate, and proportion, highlights the development of meanings of each and the construction of contextual inter-relationships among them. The lack of attention to meaning, I believe, is at the root of many problems that become visible only later in students' learning.

Unfortunately, to declare a shortage of ideas and meanings in mathematics teaching and curricula is not the same as saying what having them is like. The National Mathematics Advisory Panel final report calls repeatedly for balanced emphases upon conceptual understanding and procedural fluency. But it does not explain what conceptual understanding is or how one might teach for it. NCTM's *Principles and Standards for School Mathematics* repeatedly extols the community to have students understand the mathematics they learn so that they are better prepared to understand ideas they encounter in the future. But in not one of 857 instances in which the *Principles and Standards* uses "understand" and its derivatives does it explain what "to understand X" means. "Understand" is one of mathematics education's most primitive terms.

To make this point, that inattention to meaning is at the root of many problems of students' learning, I will develop three examples. The first will draw from trigonometry, the second from linear functions, and the third from exponential functions.

TRIGONOMETRY

Trigonometry is a notoriously difficult topic for U.S. middle-school and secondary-school students. I claim that the roots of students' difficulties lie in an incoherence of foundational meanings developed in grades 5 through 10. The U.S. mathematics curriculum develops two unrelated trigonometries: the trigonometry of triangles and the trigonometry of periodic functions.

In the trigonometry of triangles, students are taught SOH-CAH-TOA, which stands for *Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse, and Tangent is Opposite over Adjacent.* They are also taught that some triangles are special, meaning that you know their angles and their relative side lengths. The triangles are 30-60-90 degree triangles, 45-45-90 degree triangles, etc. They then are given many exercises in which they solve for the length of some missing side.

The idea of angle measure is hardly present in the standard U.S. development of triangle trigonometry. By this I do not mean references to an angle's number of degrees. Such references abound. In students' understanding sine, cosine, and tangent do not take angle measures as their arguments. Rather, they take triangles as their arguments. A question might mention a specific angle measure, but the angle having that measure is always in a triangle and the argument to the trig function involves the whole triangle. Also, angles do not vary in triangle trigonometry. In fact, students find it problematic just to imagine how an angle in a triangle might vary (Figure 1). Clearly, students who enter their study of trigonometric functions with the image of triangle variation depicted in Figure 1 will have a difficult time thinking, in any way we would find acceptable, of variable angle measures in relation to sine, cosine, and tangent.
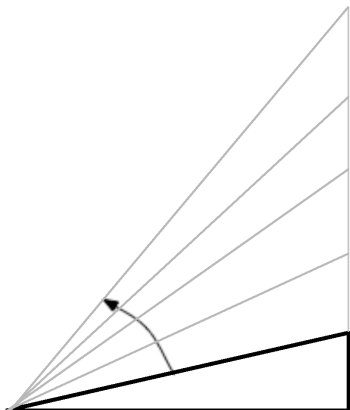
Figure 1. A common way for students to imagine varying an angle in a triangle.

Angle measure is problematic in another way. It has no clear meaning in students' thinking. This is understandable upon examining particular practices. Figure 2 shows how angle measures are often presented. The arc in the left diagram serves no other purpose than to indicate the angle upon which the text wants students to focus. It is nothing more than a pointer, which could be accomplished just as well with a different indicator (e.g., as in the right diagram of Figure 2). The right diagram highlights the shallow meaning that we unwittingly convey to students about what we are measuring when we measure an angle. "Whatever 37° means, that's

what this angle is!" As an aside, textbooks say, for example, "sin A" to indicate the ratio they want students to think of, even though "A" is not a number. It is the angle's name. This is what I meant when students understand trig ratios as taking triangles as their arguments instead of numbers.
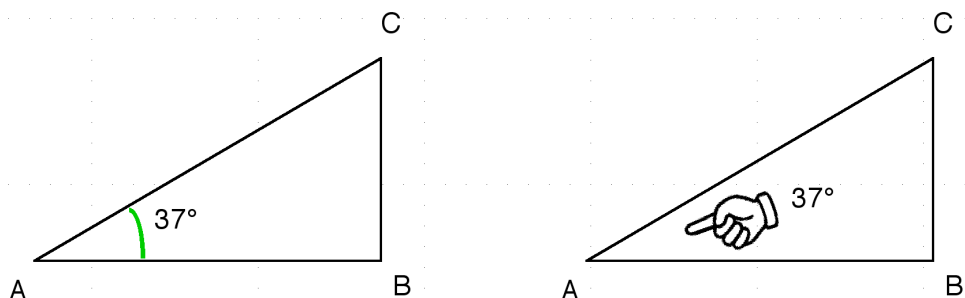
Figure 2. Conventional arc indicates the angle to which a number of degrees is attached.

This lack of clear meaning for angle measure continues from early grades through calculus. Figure 3 shows two diagrams from a popular U.S. calculus book in its development of radian measure (a book known for its "conceptual" approach to calculus). In these diagrams, "θ" is used just as is "A" in Figure 2—to name the angle. The grammar in Figure 3's right diagram even suggests this. The phrase "… spanned by θ" makes sense only if θ is the name of the angle. It is not a number. Clearly, this is a holdover of habits established in the authors' history with triangle trigonometry.
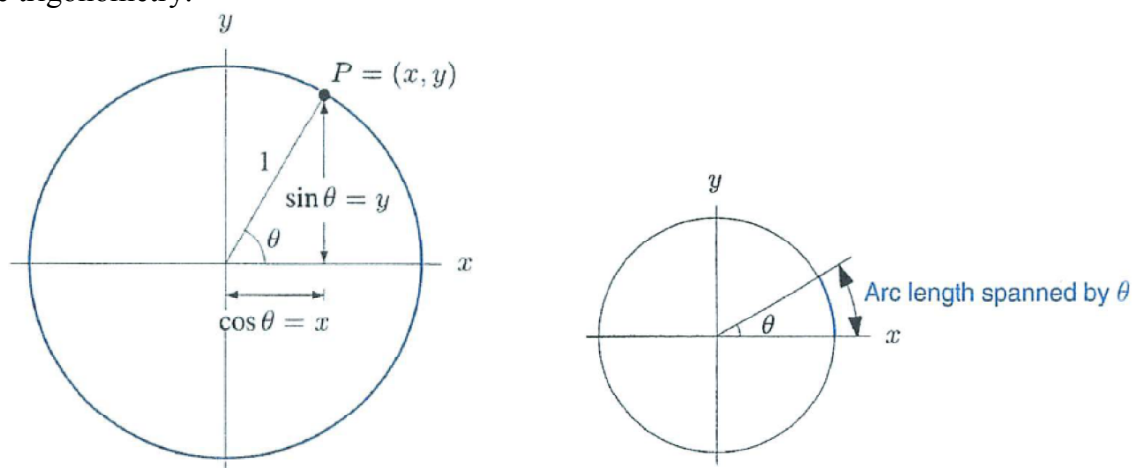
Figure 3. Figures from a popular calculus book in which θ is the name of the angle instead of representing a measure of it.

Using a letter to name an angle instead of representing its measure is embedded deeply in the school mathematics culture. In a recent professional development project, a facilitator to a school-based professional learning community suggested, in the context of the group developing a unit on trigonometric functions, that teachers ask their students to use a string to "estimate the sine of 1". The teachers looked at each other, then one asked, "What do you mean … 'the sign of 1'? That doesn't make sense." Then another teacher interjected, "Oh, you mean estimate the sine of θ where θ has a measure of 1 radian!" The facilitator asked what the difference was. The

teachers responded that the second was much clearer. Videos of their implementation of this unit confirmed that they were thinking of θ as naming the angle, not representing its measure.[2]

Calculus texts' treatments of radian measure have the intention of measuring an angle's "open-ness" by measuring the length of the arc that the angle subtends in a circle centered at the angle's vertex. The reason that 1 radius is used as the unit of measure is so that the circle's size does not affect the angle's measure. In fact, we do not need to use a radius as our unit. We could satisfy this constraint (the circle's size cannot matter) by using any unit that is proportional to the circle's circumference (Thompson, Carlson, & Silverman, 2007). The reason that we use a radius as the unit of arc length by which we measure angles is that $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ when θ is a number of radii, whereas $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = \dfrac{\pi}{180}$ when θ is a number of degrees.

Angle measure in earlier grades, when developed at all, is taught as a fraction of a complete rotation converted into an equivalent fraction of 360. To be a measure, though, we must say what about an angle we are measuring and the method by which we derive a measure of it. However, angle measure in degrees is taught as a procedure, it is not really taught as a measure. We need to develop angle measure in degrees so that it is a measure of something, and so that as a measure it coheres with radian measure. One way to do this is to base the idea of a degree also on arc length. One degree would then be an arc of a circle whose length is 1/360 the circle's circumference. The property being measured is the angle's "open-ness". The method of measuring that open-ness would be to draw a circle centered at the angle's vertex and measure the arc that the angle subtends in units of arc that are 1/360 the circle's circumference. In this way, degree measure and radian measure are exactly the same type of thing—a measure of subtended arc.

It is worth mentioning that I just outlined the principle by which a protractor works. Figure 4 illustrates this. Of course, for students to "see" an indicator arc as depicted in Figure 4, they must be taught, and must learn, the scheme of meanings behind it. Moreover, they must practice reasoning with these meanings so that those meanings become, indeed, their way of seeing measured angles. To develop relationships between degree and radian measure, they must first understand the conventionality of both, just as we expect them to understand that the relationship between measuring temperature in Fahrenheit and Celsius is between measuring the difference between freezing and boiling temperatures of water in 180 segments or 100 segments. The measured thing (an amount of arc) is the same either way; its magnitude is simply cut into different numbers of segments according to the system we happen to use (Figure 5).

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[2] It would be more accurate to say that they could think about it both ways, but in the sense that θ could represent *both* the angle and its measure.
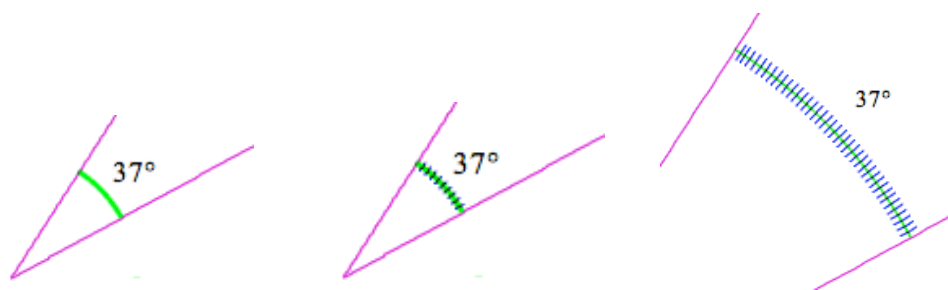
Figure 4. Seeing the "indicator arc" as a subtended arc of a circle centered at the angle's vertex, measured in a unit of arc whose length is 1/360 the circle's circumference.
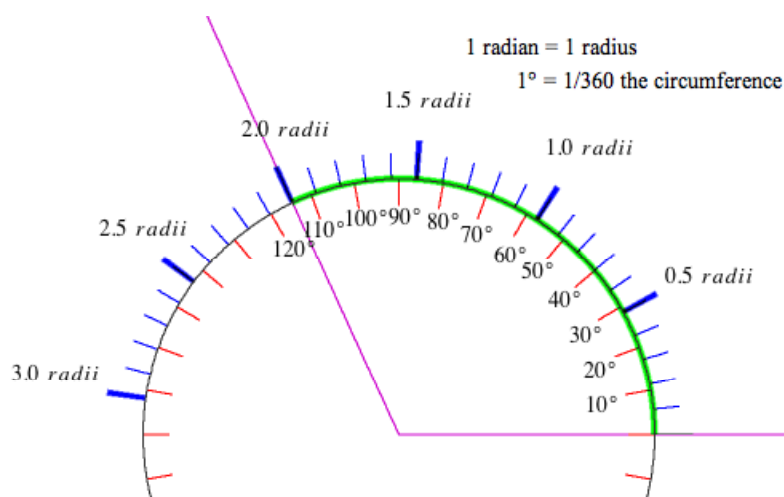
Figure 5. An angle measured simultaneously in degrees and in radians.

Curricular treatments of triangle trigonometry and of periodic functions that are coherent both within themselves and between the two developments would draw from meanings specific to each and from meanings that are common to each. Triangle trigonometry would draw from the meaning of angle measure as outlined above and would also draw from similarity –that similar triangles have the same ratios. Thus, to know the ratios between sides of one triangle will give the ratios of corresponding sides of all similar triangles. Periodic functions would draw from the meaning of angle measure as outlined here, from similarity, and would additionally highlight how one must think of varying an angle so as to systematically vary its measure. The rest is details.

I should point out that this discussion of meanings for angle measure highlights the important consideration that meanings students create at the time of learning something are highly consequential for their later learning that depends on it (Wearne & Hiebert, 1994). Students who early on learn that sinA means SOH will be at a severe disadvantage when needing to think that trig functions take angle measures as arguments. Students who learn early on that angle measures are indexical—that, for instance "90° means perpendicular" will be a severe disadvantage when angle measure needs to be thought of as a continuous variable.

LINEAR FUNCTIONS (CONSTANT RATE OF CHANGE)

The idea of constant rate of change is foundational to understanding linear functions and subordinate ideas such as average rate of change, proportionality, and slope. However, as Lobato

has shown, many students do not see ideas of rate of change, average rate of change, proportionality, and slope as being interconnected (Lobato, 2006; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002). They see them as separate sets of procedures and see them as associated with unrelated contexts. Many teachers have similar disconnections. Coe (2007) modeled three high school algebra and calculus teachers' meanings using semantic maps gained from interviews over six months. He found that all three had few connections between their meaning of slope and their meaning of constant rate of change, and they had effectively no connections among their meanings of constant rate of change, average rate of change, instantaneous rate of change, slope, and proportionality.

The problem with students and teachers' understanding of the foundations of linear functions is deeper than not having connections among meanings. The meanings they have of constant rate of change cannot provide those connections. Hackworth (1995) studied the effect of first semester calculus on calculus students' understandings of rate of change. She found, using a test-retest method, that the vast majority of 90 students (enrolled in multiple sections) had very weak understandings of rate of change at the beginning of the course and even weaker understandings at the end of the course. Average rate of change meant the arithmetic mean of the rates. Constant rate of change meant that the instantaneous rate of change did not change. But instantaneous rate of change was the number that a speedometer is pointing at were you to freeze time. In other words, students understanding of constant rate of change rarely involved two quantities changing, and certainly did not involve them changing in such a way that all changes in the value of one quantity, no matter how small or large, are proportional to corresponding changes in the value of the other.

In conversations with teachers and future teachers in many convenience samples (i.e.., students and teachers with whom I've worked), I ask them to explain the idea of average speed as if to someone who did not already understand it. The most common answer by far is, of course, "distance divided by time". I point out that this might calculate a value for an average speed, but it is not the meaning of average speed. Eventually, often with considerable support from me, they come to the meaning of average speed as entailing these aspects:

- It involves a complete trip or the anticipation of a complete trip (i.e., having a start and an end).
- The trip takes or will take a path which involves moving a definite distance in a definite amount of time.
- The average speed for that trip is the constant speed at which someone must travel to cover the same distance in the same amount of time.

But this does not answer the question, "Why divide the number of distance units by the number of time units to calculate an average speed?" To answer this question requires that we have a powerful meaning for constant speed.

One meaning of constant speed is that all amounts of distance (say, number of feet) traveled in any amount of time (say, number of seconds) are proportional to the number of seconds in which you traveled that distance. To travel at a constant speed of 88 ft/sec means that in *any* period of 1/1000 second you will have traveled 1/1000 of 88 feet. It means that in *any* period of $1/10^7$ seconds you will have traveled $1/10^7$ of 88 feet. Therefore, to say that you traveled d feet in t seconds at a constant speed means that in any one second ($1/t$ of t seconds) you will have traveled $1/t$ of the time in which you traveled d feet, and therefore you will have

traveled (1/t) of *d* feet in one second.[3] But (1/*t*) of *d* feet is the same number of feet as *d* ÷ *t*. Therefore, the constant speed, in feet per second, at which one must travel to move *d* feet in *t* seconds is calculated by *d* ÷ *t*. That is, you divide distance by time because of the proportional relationship between distance traveled and time taken to travel that distance when traveling at a constant speed.

When students understand the ideas of average rate of change and constant rate of change with the meanings described here they see immediately the relationships among average rate of change, constant rate of change, slope, secant to a graph, tangent to a graph, and the derivative of a function. They are related by virtue of their common reliance on meanings of average rate of change and constant rate of change.

Another way to view the relationship between quantities that change together at a constant rate is that there is a homogeneous relationship between the two. Kaput and West (1994) noticed this in their investigation of students' understanding of rate of change. They noted that to understand constant rate of change entails the same mental operations as understanding uniform concentration or uniform density. Harel (1994) saw the same relationship as students came to conceptualize constancy of taste with regard to anticipating differences in "oranginess" of different sized sips of a mixture in which orange pulp and water are mixed thoroughly. The idea of uniform concentration is that if you mix *m* units of substance A thoroughly with *n* units of substance B, then to say they form a uniform concentration means that any sample of the mixture will contain the two substances in the same proportion as any other sample (including the entire mixture). Similarly, if a substance has uniform density, then any part of that substance will have its volume and mass in the same proportion as any other part (including the entire amount).

This way of thinking about constant rate of change, that corresponding changes in two quantities are homogeneous, supports thinking about continuous variation of one quantity and concomitant continuous change in the other. If one quantity changes by some extremely small amount, then the other must change accordingly in the same proportion. Thus, if we work with students so that they develop a rich meaning of constant rate of change, we will at the same time support them in coming to conceptualize functional relationships as entailing continuous variation and as entailing a relationship between values that remains the same evean as the values themselves vary. Boyer (1946) anticipated this when he outlined an approach to functions that starts with proportional reasoning. Piaget and colleagues (Piaget, Blaise-Grize, Szeminska, & Bang, 1977), though uninterested in issues of teaching or curriculum, also saw proportionality at the root of ideas of function relationship. In a recent teaching experiment (Thompson, McClain, Castillo-Garsow, Lima, in preparation), a teacher who based her Algebra I instruction on ideas of constant rate of change and continuous variation led her students to think with remarkable sophistication about the behaviors of linear, quadratic, and polynomial functions and their analytic properties.

In regard to homogeneity it is worth mentioning that homogeneity is not characteristic of Confrey's notion of constant rate of change. She characterizes constant rate of change as a unit-per-unit comparison (Confrey, 1994; Confrey & Smith, 1994, 1995) and includes comparing successive values of a function (when there are such things) as the units being compared. I will

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[3] If you travel 288 feet in 7 seconds at a constant speed, then in any 1 second (1/7 of 7 seconds) you will travel 1/7 of 288 feet.

elaborate upon this in my discussion of exponential functions, but it is worth mentioning now that her notion of constant rate of change, which she devised largely so that she can say that exponential functions have a "multiplicative" constant rate of change—evaluated by $f(x + \Delta x) / f(x)$.

The value of $f(x + \Delta x) / f(x)$ for any exponential function is dependent upon the size of Δx. If the underlying function is $f(x) = 2^x$, then 2 is its constant multiplicative rate of change when $\Delta x = 1$, $\sqrt{2}$ is its constant multiplicative rate of change when $\Delta x = 0.5$, and $\sqrt[10]{2}$ is its constant multiplicative rate of change when $\Delta x = 0.1$. Different values of Δx produce a different constant rate of change for the same underlying function. Thus, constant multiplicative rate of change for exponential functions is not homogenous in Confrey's scheme. The same exponential function has different constant multiplicative rates of change depending on the granularity with which you examine changes.

Confrey's notion of rate of change has another consequence that has not been mentioned. Homogenous rate of change entails the characteristic that constant rate of change entails change and accumulation simultaneously (Thompson, 1994a; Thompson & Silverman, 2008). This was the foundation of Newton's approach to calculus and is the root idea of the Fundamental Theorem of Calculus. However, Confrey's notion of rate of change entails only the idea of change, it does not entail the idea of accumulation. As such, Confrey's system has no Fundamental Theorem of Calculus. There are no Taylor series in Confrey's system. There is no way to systematically calculate an approximate value of 2^x for non-integral values of x.[4]

Finally, continuous variation in Confrey's way of thinking about multiplicative change is very hard to imagine. If the underlying idea is that all multiplicative change happens by a split, then I do not know how to imagine the value of 2^x varying smoothly as I smoothly vary the value of x. Again, I'll return to this under exponential functions.

EXPONENTIAL FUNCTIONS

All teachers of calculus know that a defining characteristic of exponential functions is that the rate at which an exponential function changes with respect to its argument is proportional to the value of the function at that argument. A natural question is how to have this property emerge meaningfully in students' thinking. A well known approach to developing ideas of exponential function is by developing the idea of splitting (Confrey, 1994; Confrey & Smith, 1995; Smith & Confrey, 1994), where growth happens by a constant multiplier (e.g., in a geometric sequence). But the idea that an exponential function's rate of change is proportional to the value of the function does not arise easily in the splitting approach, especially if we hope to develop this idea in the context of continuous variation. I propose another approach: Start with simple interest.

The amount of money in a deposit account earning simple interest and starting with P dollars grows at a constant rate with respect to time. If the interest rate is 8% per year, then the value of the account after x years is $v(x) = P + 0.08Px$ dollars. The formula $P + 0.08Px$ makes it clear that the account's value grows at a rate that is proportional to the initial value of the deposit.

If the bank compounds interest at the end of each year, the conventional practice, and the conventional way of thinking about growth in the account's value, is that the bank adds earned

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[4] Put another way, there are no calculators in a splitting world.

interest only at the end of each compounding period, in this case at the end of each year. By this method, the account's value over time is given by the formula

$$v(x) = P(1.08)^{\lfloor x \rfloor}, 0 \le x \qquad (1)$$

where "$\lfloor x \rfloor$" means "floor x", or the greatest integer less than or equal to x. Thus, after 2.3 years the account's value will be $v(2.3) = P(1.08)^{\lfloor 2.3 \rfloor}$, or $P(1.08)^2$. That is, for every value of x between 2 and 3 (meaning, at every moment in time during the third year), the account's value will be $P(1.08)^2$. Figure 6 shows the resulting step function (the vertical segments in Figure 6 are an artifact of the graphing program's "calculator drool").
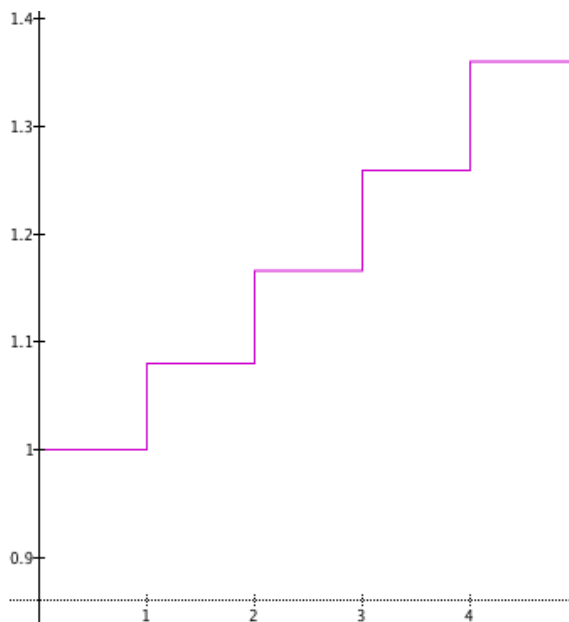

Figure 6. Graph of account value that has an initial value of $1.00 and which earns interest at 8% per year compounded annually. (Vertical line segments are just "calculator drool".)

However, we need not think of the account value as what the bank might report when you examine it online during a compounding period. This just shows the account's value at the beginning of the compounding period. Instead, we can imagine that, during each compounding period, interest accrues as simple interest. By this scheme, the accounts value grows within any compounding period at a rate of change that is proportional to the account's value at the beginning of that compounding period. The function giving the account's value by this scheme at each moment in time can be defined piecewise, as in

$$v(x) = \begin{cases} P + (0.08P)x, 0 < x < 1 \\ P(1.08) + P(1.08)(0.08)(x-1), 1 \le x < 2 \\ P(1.08)^2 + P(1.08)^2(0.08)(x-2), 2 \le x < 3 \\ \dots \\ P(1.08)^n + P(1.08)^n(0.08)(x-n), n \le x < (n+1) \end{cases} \qquad (2)$$

The difference between functions (1) and (2) is that (1) is a special case of (2). It is as if, in the case of simple interest, we are in an infinitely long compounding period.

The graph of $v(x)$ as defined in (2) appears in Figure 7. (I used an interest rate of 80% rather than 8% to accentuate the constant rate of change within compounding periods.) The function's linearity within each compounding period is a result of interest accruing at a constant rate – highlighting that the rate of change within any compounding period is proportional to the function's value at the beginning of that period.
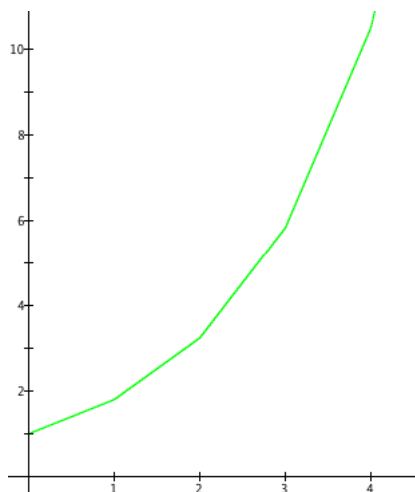
**Figure 7. Graph of $v(x)$ as defined in (2).**

Finally, we can easily adjust the definition of $v(x)$ to accommodate any number of compounding periods per year. [5] But regardless of the number of compounding periods, within any one of them, the function changes at a constant rate of change that is proportional to the value of the function at the start of that period.

Figure 8 shows three graphs, the first for 2 compounding periods per year, the second for 4 compounding periods, and the third for 12000 compounding periods (1000 times monthly)—again using a yearly rate of 80% to accentuate the change. I hope you attend to the fact that, for each graph, the function giving it increases within each interval of length $1/n$ at a rate that is proportional to the function's value at the beginning of that interval. The intent is that students come to see that, for very large n (a very large number of annual compounding periods and thus a very small amount of time), the function's value at the beginning of each period is "nearly equal to" the function's value at every point within the period. Thus, the characteristic property of exponential functions, that an exponential function always changes at a rate that is proportional to the function's value, emerges naturally from the idea of compound interest.

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[5] The function definition for the account's value, at an annual rate r and n compounding periods per year, after x years (x varying continuously) is $f(x) = P\left(1 + \dfrac{r}{n}\right)^{\lfloor xn \rfloor} + \dfrac{r}{n} P\left(1 + \dfrac{r}{n}\right)^{\lfloor xn \rfloor} \left(xn - \lfloor xn \rfloor\right)$. In this case it is still apparent that, within any compounding period, the account's value increases at a constant rate that is proportional to the account's value at the beginning of that period.
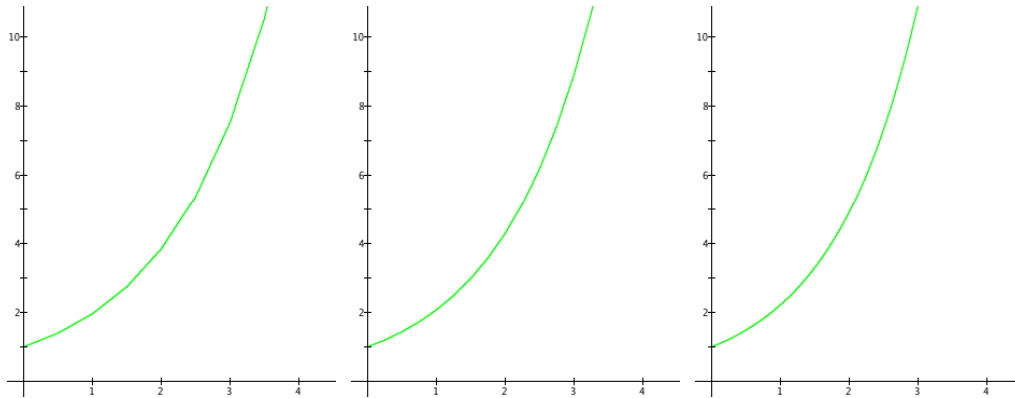
Figure 8. Graphs of *v*(*x*) for 2 compounding periods per year, 4 compounding periods per year, and 12000 compounding periods per year.

Two observations are worth making about this development of exponential function. The first is that the idea of *r* in e^{rt} being like a constant rate of change does not arise from anything special about e^{rt}. Rather, *r* having a meaning like constant rate of change arises metonymically by virtue of the fact that simple interest during any compounding period is a constant rate of change and because, regardless of the number of compounding periods, we always refer back to the annual (simple) interest rate.

The second observation is that this development accentuates the characteristic property of exponential functions (rate of change being proportional to the value of the function) at the expense of the intuition of doubling, tripling, etc. that comes from the idea of splitting. They both rest on a multiplicative conception of comparison and growth, but the two do not tie together neatly.

CONCEPTUAL ANALYSIS

The examples given above each entailed a conceptual analysis of a mathematical idea. Two issues arise immediately:

- What is conceptual analysis (and how does one do it)?
- What use is conceptual analysis for mathematics education?

We often need to describe what students might understand when they know a particular idea in various ways. Glasersfeld (1995) calls his method for doing this *conceptual analysis.* As Steffe (1996) notes, the main goal of conceptual analysis is to propose answers to this question: "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (Glasersfeld, 1995, p. 78).

Glasersfeld first introduced me to conceptual analysis when he wondered how to convey the concept of triangle to a person who is congenitally blind and does not know the word already. His example went like this (if you are sighted, close your eyes)

Imagine that you:

- are in some location, facing in some direction;
- walk, straight, for some distance.

- Stop. Turn some amount.

- walk straight for another distance.

- Stop. Turn to face your starting position.

- Walk straight to it.

Your path is a triangle.[6] (Glasersfeld & Czerny, 1979)

Glasersfeld employed conceptual analysis in two ways. The first was to generate models of knowing that help us think about how others might know particular ideas. Glasersfeld's meaning of model is very much like Maturana's (1978) notion of scientific explanation.

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena. (p. 29)

Glasersfeld's operationalization of "triangle" was more than a way to define it to a blind person. It was also an attempt to develop one hypothesis about the operational aspects of imagining a triangle. I find this approach especially powerful for research on mathematics learning. For example, in research on students' emerging concepts of rate it has been extremely useful to think of students' early understanding of speed as, to them, speed is a distance and time is a ratio (Thompson, 1994b; Thompson & Thompson, 1992, 1994). That is, speed is a distance you must travel to endure one time unit; the time required to travel some distance at some speed is the number of speed-lengths that compose that distance. Upper-elementary school children bound to this way of thinking about speed will often use division to determine how much time it will take to travel a given distance at a given speed, but use guess-and-test to determine the speed required to travel a given distance in a given amount of time. Their employment of guess-and-test is not a change of strategy. Rather, it is an attempt to assimilate the new situation into their way of thinking about speed – that it is a distance. Guess-and-test is their search for a speed-length that will produce the desired amount of time when the given distance is actually traveled.

There is a second way to employ Glaserseld's method of conceptual analysis. It is to devise ways of understanding an idea that, if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise. For example, I was working with high school students on the idea of sampling distributions. We were discussing opinion polling, and they were having difficulty distinguishing between the ideas of population parameter and sample statistic, and I began to suspect that their main problem was that they were unable to conceive a population parameter. I found myself saying this:

> Suppose we are like Mork[7] and can stop time for everyone but ourselves. Imagine freezing everyone in our target population.

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[6] For readers who recall Logo, the similarity between Glasersfeld's operationalization of a triangle and a turtle-procedure for drawing one is striking. However, his example predates the general availibility of Logo, and neither of us had heard of it anyway.

At that moment, each person in the population has an answer (yes,
no, or no opinion) to the question we will ask, even if we happen
not to ask him or her the question. So, the population as a whole, at
that moment in time, has a percent of it who *would* say "yes" to
our question were they to be asked.

In other words, in order to talk about population parameters, students needed to think of
populations as having characteristics whose measures have specific values at each moment in
time. This is not to say that this example's population *really* had a characteristic whose measure
had specific values at each moment in time. For the purpose of building a concept of sampling
distribution, it is merely useful to think that it does. However, this was my realization – that it
was merely *useful* to think of a population having a particular measurable characteristic. Students
needed to *believe* that populations *can* have measurable characteristics, or else they would have
been unable to conceive of sampling distributions as arising from repeatedly drawing samples of
a given size from that population. They also would have been unable to consider how the set of
sample statistics clusters around the population parameter. To coordinate all these aspects of
sampling distributions, population parameters needed to be real to them.

Steffe and Tzur (Steffe, 1993; Tzur, 1999) have employed this use of conceptual analysis
to guide their instruction in teaching experiments on rational numbers of arithmetic. Confrey and
her colleagues have employed conceptual analysis in similar ways to convey how one might
think about multiplication so that it will simultaneously support thinking about exponential
growth (Confrey, 1994; Confrey & Smith, 1994, 1995). Thompson & Saldanha (2003) employed
conceptual analysis to show how a person's understandings of multiplication, division,
measurement, and fraction could each be expressions of a core scheme of conceptual operations,
all entailed by multiplicative reasoning. As Steffe (1996) noted, conceptual analysis (the
conjoining of a theory of mathematical understanding and radical constructivism as an
epistemology) emphasizes the positive aspect of radical constructivism – that knowledge persists
because it has proved viable in the experience of the knower. Knowledge persists because it
works.

Conceptual analysis can also provide a technique for making operational hypotheses
about why students have difficulties understanding specific situations as presented in specific
ways. For example, standard fractions instruction often proposes fractions as "so many out of so
many" (e.g., 3/5 of 10 apples is "three parts out of five equal-sized parts of the 10 apples". When
students understand fractions, in principle, as "so many out of so many", they understand
fractions as an additive part-whole relationship. Fractional relationships like "7/5 of 10" apples
make no sense whatsoever to students who understand fractions additively, because they would
have to understand it as specifying "seven parts out of five equally-sized parts of 10 apples".

Finally, as illustrated in this paper's first part, conceptual analysis can be employed to
describe ways of understanding ideas that have the potential of becoming goals of instruction or
of being guides for curricular development. It is in this regard that conceptual analysis provides a
method by which to construct and test a foundation of mathematics education in the same way
that people created a foundation of mathematics.

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[7] Of Mork and Mindy, a television program of the 1970's, starring Robin Williams, about an alien living on earth.
Many students had watched reruns of this program.

In summary, conceptual analysis can be used in four ways:

(1) in building models of what students actually know at some specific time and what they comprehend in specific situations,

(2) in describing ways of knowing that might be propitious for students' mathematical learning, and

(3) in describing ways of knowing that might be deleterious to students' understanding of important ideas and in describing ways of knowing that might be problematic in specific situations.

(4) in analyzing the coherence, or fit, of various ways of understanding a body of ideas. Each is described in terms of their meanings, and their meanings can then be inspected in regard to their mutual compatibility and mutual support.

I find that conceptual analysis, as exemplified here and practiced by Glasersfeld, provides mathematics educators an extremely powerful tool. It orients us to providing imagistically-grounded descriptions of mathematical cognition that capture the dynamic aspects of knowing and comprehending without committing us to the epistemological quagmire that comes with low-level information processing models of cognition (Cobb, 1987; Thompson, 1989). Conceptual analysis provides a technique for making concrete examples, potentially understandable by teachers, of the learning trajectories that Simon (1995) calls for in his re-conceptualization of teaching from a constructivist perspective, and which Cobb and his colleagues employ in their studies of emerging classroom mathematical practices (Cobb, 2000; Gravemeijer, 1994; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). In addition, when conceptual analysis is employed by a teacher who is skilled at it, we obtain important examples of how mathematically substantive, conceptually-grounded conversations can be held with students (Bowers & Nickerson, in press). Teachers in the U. S. rarely experience these kinds of conversations, and hence they have no personal image of them. Having positive examples of such conversations will be very important for mathematics teacher education.

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