Weber, E., Tallman, M., Byerley, C., & Thompson, P. W. (in press). Introducing derivative via the calculus triangle. *Mathematics Teacher.*

Introducing Derivative via the Calculus Triangle

Typical treatments of derivative do not clearly convey that the derivative function represents the original function's rate of change. We argue that revealing the relationship between a function and its rate of change function for static values of x does not facilitate productive ways of thinking about generating the rate of change function or allow students to anticipate the graphical behavior of the rate of change function through examining a graph of the original function. Accordingly, we propose an approach that builds upon Thompson's (1994, 2008) calculus research that introduces derivative in a way that maintains the centrality of rate of change as a conceptual underpinning of derivative. In this section we explain the *calculus triangle* approach and illustrate how the approach facilitates mature understandings of derivative by providing examples of the approach's utility in novel and routine settings.

Our group of teacher-researchers designed and taught two university calculus courses that emphasized rate of change and quantitative reasoning. Our approach proposed the concept of a *calculus triangle* to support students in attending explicitly to quantities, and constructing a method for creating and tracking the ratio of changes in quantities to produce a rate of change function. We have found that the calculus triangle allows students to reason flexibly across mathematical domains such as differentiation, accumulation, as well as across graphical representations.

Observations and Research about Students' Understanding of Derivatives

Historically, the derivative was constructed as a way to represent and measure the rate at which one quantity changes with respect to another quantity. However, many students are taught in a way that enables them to solve calculus problems without attending to rates of change. Carlson and colleagues (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) found that second semester calculus students were unable to produce a qualitative graph that expressed the height of water in a bottle as a function of the water's volume. The students used memorized properties of second derivatives but could not relate inflection points in the graph of the function to changes in width of the bottle. Additionally, a number of authors reported students' difficulties in creating graphical representations of a function's rate of change function (Tall, 1986; Ubuz, 2007). These researchers found that students often focused on computing derivatives without connecting the derivatives they computed and evaluated to a function's rate of change at specific points in its domain (Ubuz, 2007).

We Know Derivatives Are About Rates. Why Don't Our Students?

Calculus books' attention to velocity and development of the limit definition of the derivative suggest textbook authors intend to develop the idea of derivative as a rate of change. We have focused heavily on how students think about derivatives to understand the disconnect between what the textbooks present and what students understand. In our survey of best selling calculus books in the United States, we found the textbooks qualitatively generated the derivative function in different ways. In most books, the secant does not slide through the function's domain. Rather, one intersection of the secant line slides toward the other intersection, creating successively better approximations of the tangent line. The tangent line, however, slides through the function's domain. Ferrini-Mundy and Graham (1991) found that students often struggled to envision and make sense of a sliding secant line and its relationship to rate of change on a small interval and believed the secant line collapsed into a single point to create the sliding tangent. Our observations suggest that this problem persists today.

Even if students were successful in envisioning the tangent line as representing the rate of change at a point they commonly confounded the notion of derivative at a point with the derivative function (Ubuz, 2007). Many calculus books do not explicitly describe how to think about rate of change on one small interval to support constructing a function that gives the original function's rate of change over its domain. We did not locate any textbooks that helped students think about rate of change over small intervals as generalizing to rate of change over the function's domain.

We not only observed the difficulties documented by Ubuz and Ferrini-Mundy, but also found that it was non-trivial for students who primarily remembered slope as "rise-over-run" to think of difference quotient as a rate. After discussing the ratio $\frac{f(x+h)-f(x)}{h}$ with our students it was apparent that many of them did not see $f(x+h)$ and $f(x)$ representing amounts of a quantity associated with particular inputs. Without an understanding of $f(x)$ as giving the value of a quantity they did not see the $f(x+h)$-$f(x)$ as how much the quantity changed in relation to a given change in input.

We conjecture that students' difficulties with function notation, their struggle to connect algebraic and graphical representations of functions, and understandings of rate of change may explain their struggle to think about derivative as a function. We believe that thinking about derivative as an object of calculation may be attributed in part to the students' lack of attention to and construction of quantities in a way that would allow them to track the ratio of the quantities' changes. Given this hypothesis, we have attempt to facilitate productive mental images of the derivative as a function whose values give the rate of change of another function *f* by continuously tracking the average rate of change of Quantity A ($f(x)$) with respect to Quantity B (x) over a continuum of values for Quantity B (x to $x+h$). We use the mathematical construct of a "calculus triangle" to achieve this coordination.

The Calculus Triangle

The intent of the calculus triangle approach is to allow students to envision change in a function due to change in its argument. However, there is a distinct danger that (1) students will see a calculus triangle as a geometric object, and (2) students will see only one calculus triangle at a time. Regarding the first possibility, we want to students to see the "legs" of the triangle as changes in input and output of a function and the "hypotenuse" as the graph of a linear function (see Figure 1). Regarding the second concern, we want students to look at the graph of any function and envision *many* possible calculus triangles. That is, we want them to see that there is a calculus triangle at *every* point on a function's graph and that these triangles can be as small as one desires (see Figure 2). To envision this possibility, we introduce the idea of a "sliding" calculus triangle. "Sliding" is produced by fixing the change in the input and allowing the input to vary through the domain of the function in a *systematic* way (e.g. left to right). The mental image that "sliding" is intended to promote is that of a calculus triangle traversing along the function.

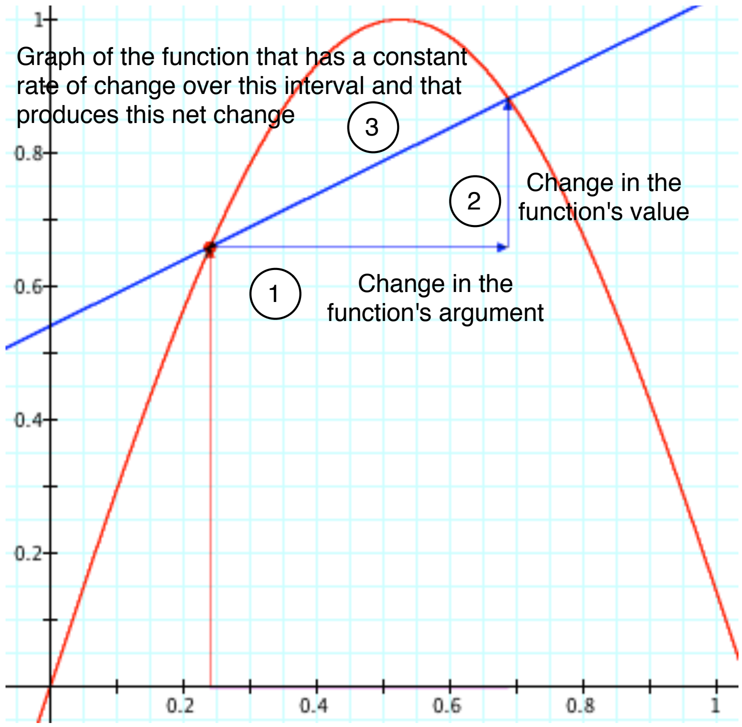

Figure 1. A calculus triangle in rectangular coordinates.

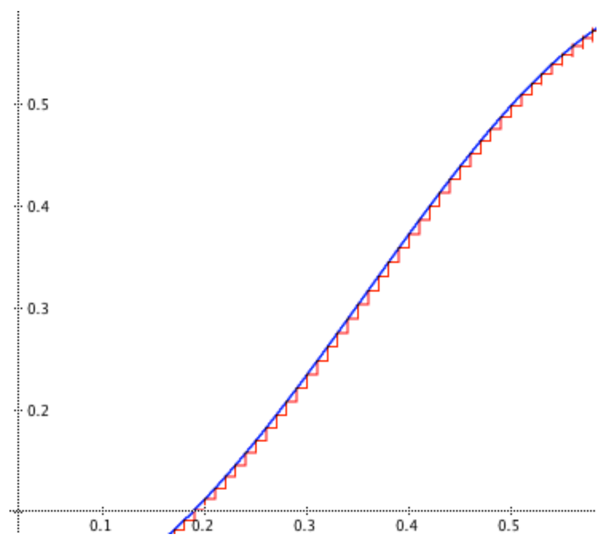

Figure 2. There is a calculus triangle at *every* point on a function's graph and that these triangles can be as small as one desires.

The definition of derivative, as it was found in the contemporary calculus books we surveyed, failed to convey mental imagery that would support students in constructing the derivative function. Recall the typical definition of derivative:

Definition: Let *f* be a function. We define the ***derivative of f at x***, *f'*, by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, provided the limit exists.

In this definition, *h* varies while *x* remains fixed. Hence, $f'(x)$ represents a scalar quantity for a specific value of *x*. Students are expected to understand that $f'(x)$ represents a function by imagining *h* approaching zero in the difference quotient for all values of *x* in the domain of *f*. Imagining varying *h* while *x* remains fixed does not allow one to visualize the derivative function being generated for a continuum of input values. In addition, by letting *h* approach zero, thus producing a difference quotient that represents the slope of a tangent line, and visualizing the tangent line sliding along the surface of the graph, it appeared to our calculus students that we were focused on the properties of this sliding tangent, when we were actually focused on the ratio of two quantities which we represented by the sliding tangent.

By attending to the sliding tangent as an object without reference to the change in quantities it represents, students often did not understand that derivative represented a rate of change because their conception of rate of change was associated with steepness of the tangent line, not comparison of changes in quantities. As an alternative to the typical definition of derivative, we avoid allowing the value of *h* in the difference quotient approach zero. Instead, we define the rate of change function r_f as,

$$r_f(x) = \frac{f(x+h) - f(x)}{h}$$

for a small, but fixed, value of *h*. Making *h* fixed but sufficiently small allows one to let *x* vary, and coordinate the corresponding variation in r_f. This simultaneous variation of *x* and r_f is illustrated in what we term the *sliding calculus triangle*. We emphasize that the calculus triangle is not a geometric object. Instead, it is a way to help students focus explicitly on changes in quantities represented by the "legs" of the triangle, the ratio of which is represented by the slope of the hypotenuse. Though we found that recognizing r_f as a function was non-trivial, that alone is often not sufficient to imagine how the rate of change function is generated. For this, we turn to the graphical representation of the rate of change function (see Figure 3).

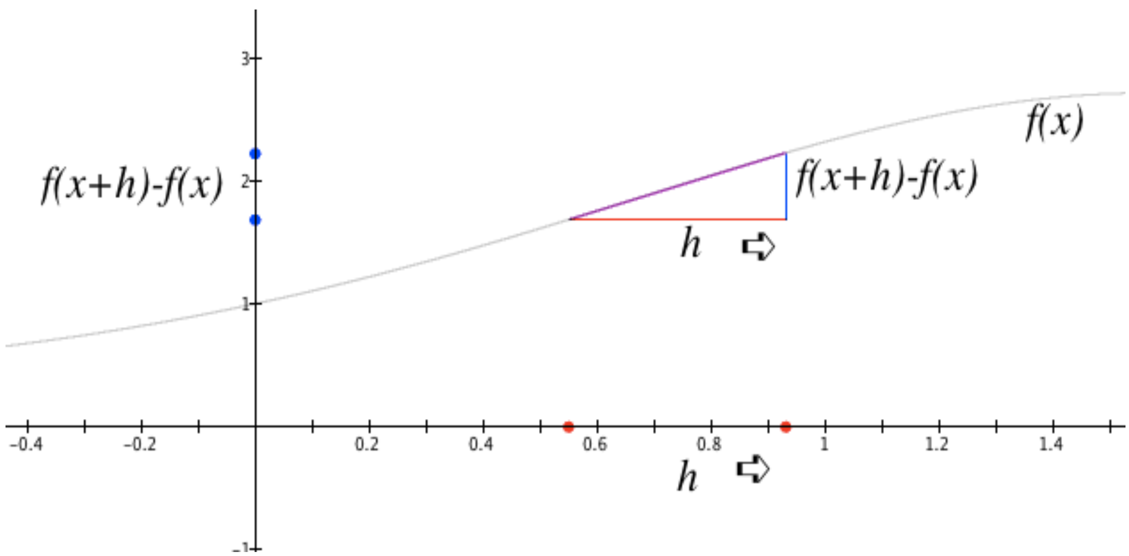

Figure 3. The interval of fixed length *h* represented on the *x*-axis slides through the domain of the function, tracking the quantities *h* and $f(x+h) - f(x)$.

In order to generate outputs for the rate of change function, we measure the quantity $f(x+h) - f(x)$ in units of *h* and systematically associate this output value with *x*, the left endpoint of the interval $[x, x + h]$. Accordingly, a point on the rate of change function can be interpreted as (x, $f(x+h) - f(x)$ units of h) (see Figure 3). Then, as *x* varies throughout the domain of the function, this point traces out the rate of change function $r_f(x)$ (see Figure 3).

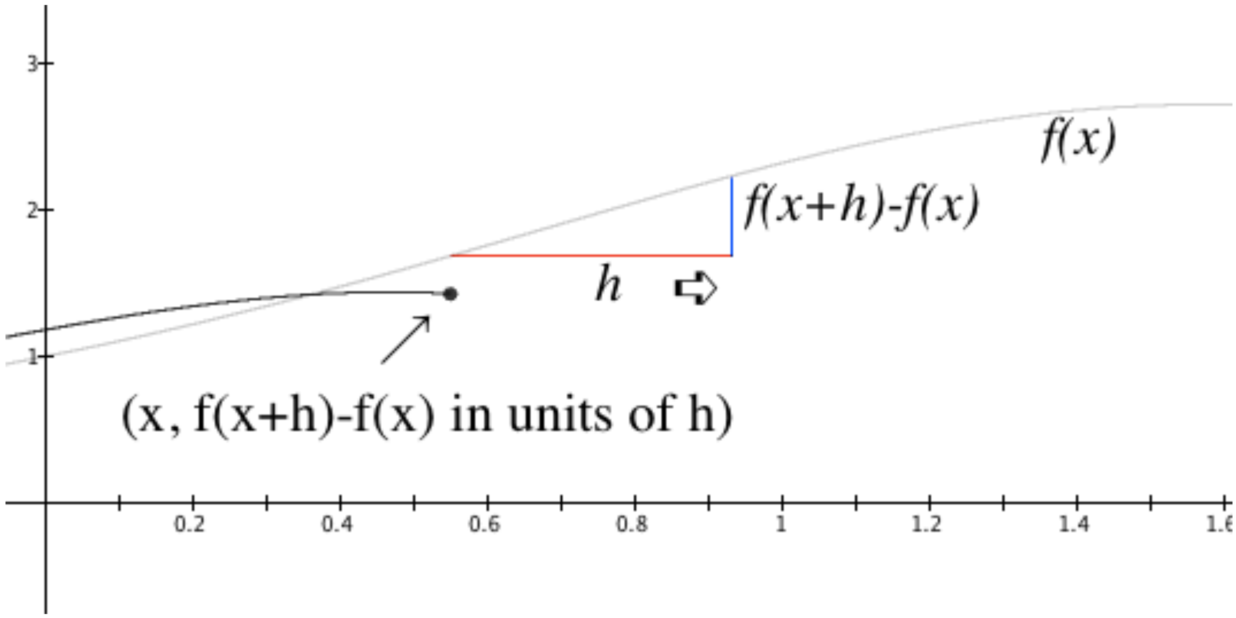

Figure 4. The calculus triangle slides as the fixed interval h slides through the domain of the function, generating the rate of change function.

It is important to note that one can slide the calculus triangle through the domain of f while coordinating the outputs being generated by measuring $f(x+h) - f(x)$ in units of h, which allows the student to think about the rate of change function being generated as the calculus triangle moves along the original function f. The mental coordination of imagining the rate of change function being generated as one traces along the function is not possible when trying to attend to two varying both h and x.

We recognize that it is arguably unsuitable to compare the calculus triangle approach with the traditional approach because the objective of the traditional approach is to make sense of the graphical representation of the *exact* derivative function (at a point) whereas the calculus triangle approach merely produces an *approximation* to the derivative function. We believe, however, that generating an approximation to the derivative function is unproblematic if an instructional effort is made to discuss the convergence (uniformly) of the approximation to the exact derivative function as h

approaches zero. The important feature distinguishing the calculus triangle approach from the traditional approach is that generating the derivative function *precedes* letting h approach zero in the calculus triangle approach whereas in the traditional approach, the limiting process comes first. We believe that when focusing on the calculus triangle, students are better suited to think about infinitesimal rate of change near a point. This approach contrasts with thinking about rate of change at a point, where quantities are not changing, and thus, discussing rate of change becomes problematic. A classroom discussion about the relative accuracy of the approximation for sufficiently small values of h can be framed so that students' reasoning relative to generating approximations of the actual derivative functions can be isomorphic to more formal definitions of uniform convergence.

Implications and Conclusion

The traditional approach to developing the derivative function can accomplish one of the following two aims, but not both: (1) derivative functions are fundamentally about rates of change, and (2) the derivative function can be generated by developing a ratio of the changes in the output quantity measured in units of the input quantity through the domain of the function f. The calculus triangle approach supports students in establishing connections between average rate of change and the derivative function while not compromising the potential for the derivative function to be generated by explicitly measuring the rate of change *of* $f(x)$ with respect to x.

Traditional treatments of calculus use the definition of derivative in an axiomatic approach to develop further rules of differentiation. This process emphasizes derivative as an operator on functions instead of emphasizing the centrality of rate of change. We believe that the calculus triangle approach allows students to understand the derivative function as tracking the ratio between changes in two quantities. If students are able to attend to derivative as a rate of change, we believe this supports them in understanding the process of differentiation as an operator that measures a rate of change.

Single and multivariable calculus, and differential equations necessitate that one perform symbolic computations as well as interpret the meaning of those computations. The calculus triangle approach supports the student in developing both computational fluency and interpreting the results of those computations meaningfully. As students approach novel problem solving situations, they are equipped with ways of thinking, particularly thinking about the derivative function as a rate of change, which are necessary to reason through these situations. By thinking about the meaning of the computations, students are able to draw connections between ideas in calculus that are typically presented as disparate because they require different techniques for differentiation.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Ferrini Mundy, J., & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly, 98*(7), 627-635.

Tall, D. (1986). Building and testing a cognitive approach to the calculus using interactive computer graphics. Unpublished Dissertation. University of Warwick.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229–274.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Washington, DC: Mathematical Association of America.

Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: Stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology, 38*(5), 609-637.