# From a Historical Observation to a Theory of Calculus Education

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## From a Historical Observation to a Theory of Calculus Education

It all started with the question, "How might Newton have understood a function's graph so that he saw the Fundamental Theorem of Calculus when viewing it?" Puzzling about that question led to a convergence of radical constructivism, quantitative reasoning, and insights into students' understandings of variation and covariation that eventually formed a theory of calculus learning and teaching.

Keywords: calculus, accumulation, rate of change, conceptual analysis, Newton, DNR

It was a morning in Summer, 1991. I stared at a computer screen showing a graph of a function. I asked myself, "How might Newton have understood this graph so he SAW the Fundamental Theorem of Calculus?" This was after reading Barron's comment that while Leibniz proved the FTC, Newton *began* with the FTC (Barron, 1969, p. 191). It took 20 years of reflection on connections among ideas of quantity, variation, covariation, and function to fully understand it.

Later that day I had one of my frequent conversations with the late Jim Kaput. I asked, "Jim, is it the case that the function $f$ in $\int_a^x f(t)dt$ is always a rate of change function? Jim replied, "Gee. I don't know. I've never thought about that." I had no idea that this question would be the key to rethinking calculus and its teaching.

The inspiration for my question came largely from two sources. The first was from David Tall, who proposed that we help students understand a function's differentiability by understanding its graph as being "locally straight" (Tall, 1996). But Newton always spoke of fluents (flowing quantities) varying by way of their fluxions (rates of change) without reference to graphs or coordinate systems. I therefore re-phrased David's notion of *locally straight* as *locally constant rate of change.* Rephrasing "locally straight" this way removed the matter from interpreting graphs to the realm of reasoning about relationships between quantities as their values varied. I then puzzled about how to connect the ideas of fluent (an amount) and fluxion (a rate) so that the two produce an accumulating amount. I outlined a conceptual analysis of Newton's thinking in Thompson (1994) and investigated difficulties students encountered in their attempts to see the FTC as stating a relationship between accumulation and its rate of change. Two major sources of students' difficulties were their impoverished meanings for how quantities vary and for rate of change of one quantity with respect to another.

The second source of inspiration was my puzzlement about how to use a Riemann sum to represent an accumulating amount. In my experience Riemann sums were always used to represent a fixed, static amount. One way to accommodate a varying upper limit would be to fix a value of $N$ and partition the varying interval $[a, x]$ always into $N$ segments of length $(x - a)/N$. This would be like having $N$ subintervals that all stretch as the value of $x$ varies. This was unsatisfactory for three reasons: (1) Approximations of accumulated totals become more inaccurate with larger variations in $x$, and integration could only be interpreted as adding up pieces. (2) The ideas of amount and rate of change of amount are largely disconnected. (3) I could not imagine this modeling any quantitative situation.

Another way to have a varying Riemann sum would be to use a fixed amount of change in $x$ and to include $x$ and $\Delta x$ in the upper limit of the summation. Then the number of subintervals varies instead of the length of each subinterval. The upper limit of the summation would be the number of intervals of size $\Delta x$ contained within the interval $[a,x]$, or $\lfloor\frac{(x-a)}{\Delta x}\rfloor$, the summation then being the approximate accumulation function $A$ defined as[1]

[1] This is assuming $x \geq a$.

$$A(a, x, \Delta x) = \sum_{k=1}^{\lfloor \frac{(x-a)}{\Delta x} \rfloor} f(a + (k-1)\Delta x)\Delta x.$$

This formulation of the approximate accumulation function *A* forces us into a particular interpretation: As the value of *x* passes through the $k^{th}$ Δ*x*-interval the value of *A* is constant. When the value of *x* reaches the end of the $k^{th}$ Δ*x*-interval, the value of the accumulation changes by $(f(a + (k-1)\Delta x) \cdot \Delta x)$.

Figure 1 shows the graph of $y = A(0, x, 0.1)$ as an approximation to the distance a ball has fallen as it speeds up according to the force of gravity on Earth—9.8 (m/sec)/sec, and 0.1 second is the change in time after which we update the approximate accumulation of distance fallen. When $k = 12$ we are in the interval from $x = 1.1$ to $x = 1.2$, so the velocity during this interval is taken to be *f* (1.1), or 10.78 m/sec, and the model proposes that the object maintains this speed for 0.1 seconds. So the additional distance fallen at the end of this interval is $(10.78 \cdot 0.1)$ meters, or 1.078 m and the total distance fallen at the end of this interval is approximately $A(0,1.2,0.1) = 5.39$ meters.

This method of approximating an accumulation function will always produce a step function. This makes sense because as the value of *x* passes through an interval $[(k-1)\Delta x, k\Delta x]$, nothing gets added to the prior accumulation until $x = k\Delta x$, so the value of *A* does not change for values of *x* *within* the interval $[(k-1)\Delta x, k\Delta x]$. This means the rate of change of approximate accumulation *within* every Δ*x*-interval is 0. It would be far more satisfactory to have the approximate accumulation function change

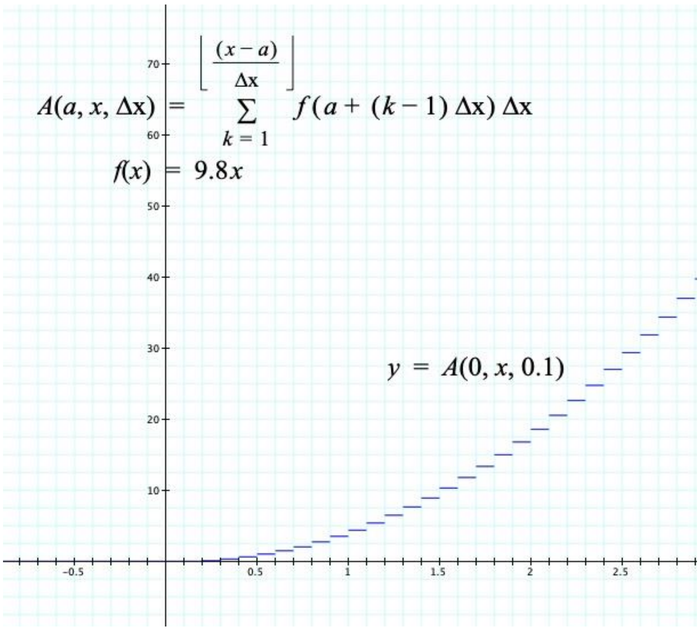


Figure 1. Graph of $y = A\ (0, x, 0.1)$

smoothly within each $\Delta x$-interval to reflect that the object is falling continuously as time elapses continuously. This would align with Euler's and Leibniz' image that graphs of continuous functions are composed of straight line segments of infinitesimal length and align with Newton's canonical vision[2] that accumulations always vary at some non-zero rate of change over an interval of change. We can capture smooth change by representing "accumulation so far" within the $\Delta x$-interval containing the current value of $x$. The question is at what rate is $A$ changing with respect to $x$ within a $\Delta x$-interval?

It was in pondering at what rate $A$ changes with respect to $x$ I realized how Newton could see the FTC while looking at a graph. If we use the "bounded area" metaphor as the meaning of an integral, and envision the value of $x$ passing through a $\Delta x$-interval, it becomes clear that the value of the function at the beginning of an interval is the rate of change of area within that rectangle with respect to $x$ as the value of $x$ varies within that interval (Figure 2). When $\Delta x$ has an infinitesimal value, the value

[2] I say "canonical" because he considered the case of no change within an interval to be degenerate, in the same way as the degenerate case of a square with side length zero is a point.

of *f*(*x*) is the value of the rate of change of accumulating area with respect to *x* at every value of *x*.

The linearization of *A* is now possible. We know the value of the function *f* at the beginning of a Δ*x*-interval is the value of the accumulation's rate of change at the beginning of that interval. The rate of change of accumulation at the beginning of a Δ*x*-interval is simply $f(\mathit{left}(a, x, \Delta x))$,[3] where the function *left* is defined as

$$\mathit{left}\ (a, x, \Delta x) = a + \lfloor\frac{(x-a)}{\Delta x}\rfloor \Delta x,$$

which is the left end of the Δ*x*-interval containing the current value of *x*, which, is *a* + (the number of complete Δ*x*-intervals from *a* to *x)* times (the length of each Δ*x*-interval).

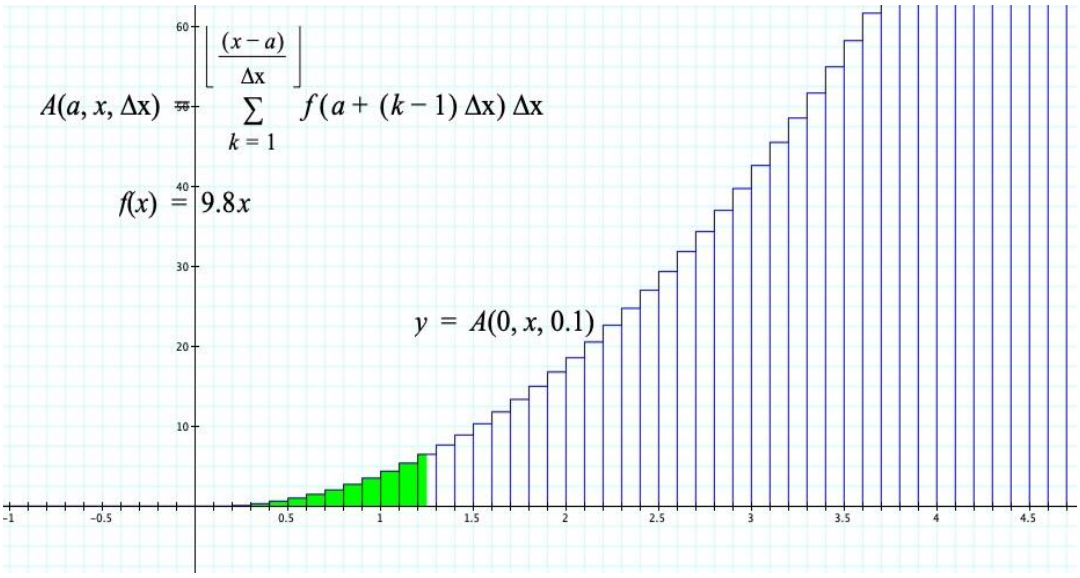


Figure 2. Determining the rate of change of accumulating area as the value of *x* varies within a Δ*x*-interval

The approximate accumulation function $A_f$ which linearizes the approximation over each interval is then (accumulation over completed Δ*x*-intervals) plus (accumulation so far within the current Δ*x*-interval), or

[3] The inclusion of *a* and Δ*x* as inputs to *left*, and to *r* and $A_f$ (below) is so that you can display multiple graphs with different parameter values simultaneously within the same coordinate system. Were values of *a* and Δ*x* defined outside the definitions of *left*, *r*, and $A_f$, you could graph just one function at a time.

$$\begin{aligned} A_f(a, x\Delta x) &= A(a, x, \Delta x) + r(a, x, \Delta x)(x - \mathit{left}(a, x, \Delta x)) \\ r(a, x, \Delta x) &= f(\mathit{left}(a, x, \Delta x)) \end{aligned}$$

Figure 3 shows three graphs on two sets of axes. The first (left axes, red) is of the falling ball with accumulation updated every 0.5 seconds but without continuous accumulation throughout the interval. It is a step function, as explained earlier. The second (left axes, blue, overlayed the first graph) is of the falling ball with accumulation updated continuously at a constant rate through each Δ*x*-interval of 0.5 seconds. This illustrates the linearization of approximate accumulation over each interval. The third graph (right axes) shows the accumulation updated every 0.1 seconds. You should note that while the graph on the right appears to be smooth, it is actually linear over every Δx-interval of length 0.1 seconds.

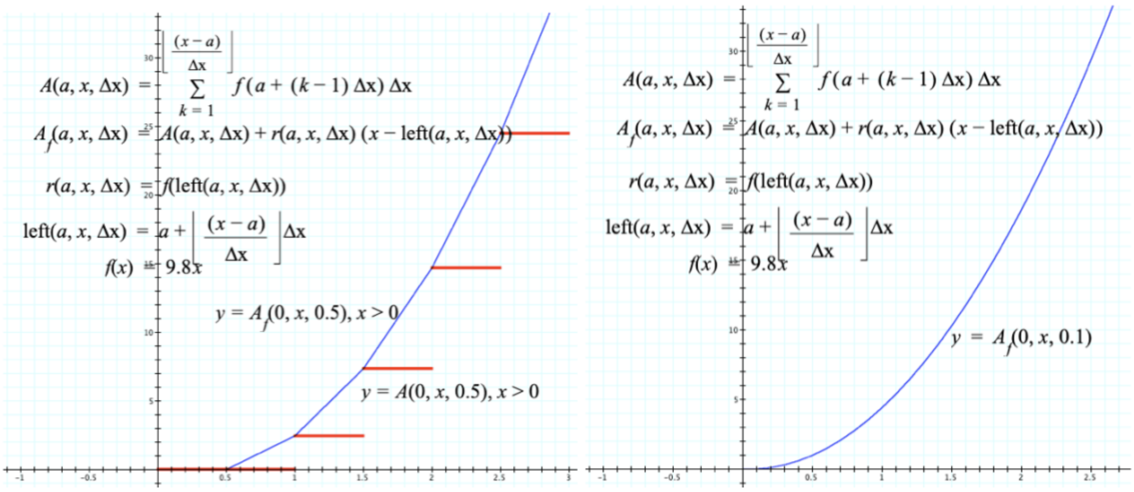


Figure 3. Approximate accumulation function linearized over Δ*x*-intervals

One last conceptual development is required to move from $A_f(a, x, \Delta x)$, an approximate accumulation function, to $F(x) = \int_a^x f(t)dt$, an exact accumulation function, while retaining the idea that *x* varies smoothly and continuously and that *f*(*t*) is the rate of change of accumulation for values of *t* from *a* to *x*. Some instructors in the development phase of DIRACC said, "When Δ*x* becomes infinitesimally small, then

'$\Delta x$' becomes '$dx$' and '$\sum$' becomes '$\int$ '", focusing on the summation part of $A_f$ as becoming the exact integral. This explanation of the transition from approximate to exact accumulation function, however, has two drawbacks: (1) it violates the commitment to the idea that the value of *t* varies continuously from *a* to *x* by varying smoothly and continuously though intervals of size Δ*x;* and (2) with the explanation offered by these instructors, bits of accumulation no longer accumulate at a constant rate of change and thus we lose the local linearity (locally constant rate of change) that motivated this approach in the first place.

To counter thinking about Δ*x* magically becoming *dx* when Δ*x* is infinitesimal I introduced the idea of *moments* of variation by way of having students examine actual motion as captured by a camera (Thompson, Ashbrook, & Milner, 2019, Section 4.3). A series of photos of cars passing a spot in a road shows that the picture is blurry no matter how fast the shutter speed (how small the amount of time the shutter is open) to convey the mantra, "all motion (all variation) is blurry." Behind this mantra is the mathematics of non-standard analysis, wherein every hyper-real number is a real number plus or minus an infinitesimal number. With this idea we can say, "As the value of *t* varies from *a* to *x* (through the reals), *dt* varies through an infinitesimal interval surrounding the value of *t*. Thus, in the exact accumulation function, $dt = (t \pm \epsilon)$, where *t* is a real number and $\epsilon$ is a variable infinitesimal. Then $f(t)dt$ retains the linearization of the accumulation function over infinitesimal intervals containing the value of *t.* The local linearity of the exact accumulation function is retained.

An important consideration of this approach to create integrals as accumulation functions is that it provides an opportunity to *necessitate,* in the sense of Harel (2013), the idea of average rate of change as the foundation for derivatives. An example: The function *f* defined as $f(x) = x^2$ gives the value of the area enclosed by a square with side-length *x*. Every value of *f* gives an amount of area. So *f* (*x*) is the area enclosed by an *x* by *x* square. However, we can also re-conceive any area bounded by a square of side length *x* as having accumulated. In other words, we have $x^2 = \int_a^x r(t)dt$ for some rate of change function *r*. We can approximate this function *r* by looking at average rates of change of $x^2$ with respect to *x* over small (infinitesimal) intervals. The derivation of *r* resembles the common derivation, but the motivation is different than the usual motivation. We are deriving a rate of change function that will give $f(x) = x^2$ as its accumulation function. In principle, we can re-conceive any function that gives an amount of a quantity as an accumulation function so that any amount accumulates at some rate of change over intervals of its independent variable. This way of thinking is common in scientific applications. We also have set the foundation for the concept of antiderivative. An *antiderivate* of the function *f* is an accumulation function that has *f* as its rate of change function. This also answers my question to Jim Kaput about whether *f* in $F(x) = \int_a^x f(t)dt$ is always a rate of change function: It is when you retain the meaning of *F* as being an accumulation function. Then *f* (*x*) is the rate of change of *F* at each value of *x.*

As an example of re-conceiving an amount as having accumulated, consider approximating the volume of a solid of revolution. I develop this approach more expansively in Chapter 8 of Thompson et al. (2019).

We first need to think of volumes of solids in a new way via a two-step process:

1. Think of the solid as having an empty shell that bounds a region in space
2. Fill the shell in a way that supports quantifying the filled region's volume. (Thompson et al., 2019, Section 8.3)

Figure 4 shows the graph of *y* = sin(*x*) from 0 to π in the midst of rotating about the *y*-axis. The graph is sprinkled with “pixie dust” so every point on it leaves a trace of where it has been. The effect of rotating the graph is to create an empty shell. The idea of approximating the volume of the bounded region is to “fill” the shell in a way that quantifies the volume of the region. The method to fill it is with cylinders that vary in volume at a known rate of change over intervals of the accumulation’s independent variable.

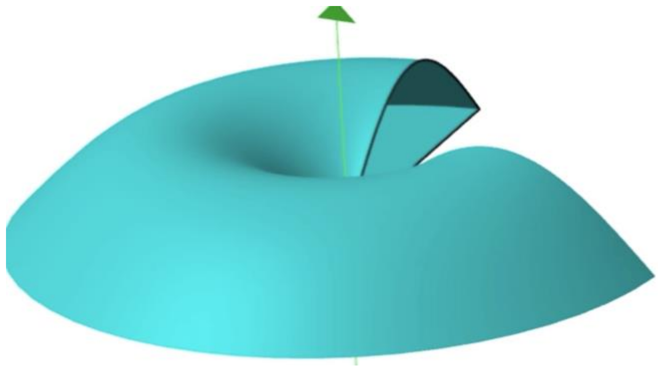

Figure 4. A shell created by rotating the graph of $y = sin(x)$,$0 \leq x \leq \pi$ and the interval $0 \leq x \leq \pi$ about the $y$-axis

There are two ways a cylinder can vary in volume (Figure 5): (1) varying height with constant base, and (2) varying radius with constant height. In (1) the cylinder’s volume varies at a constant rate *equal in value* to the area of its base, and (2) the cylinder’s volume varies at a rate *equal in value* to the cylinder’s lateral surface area.[4]

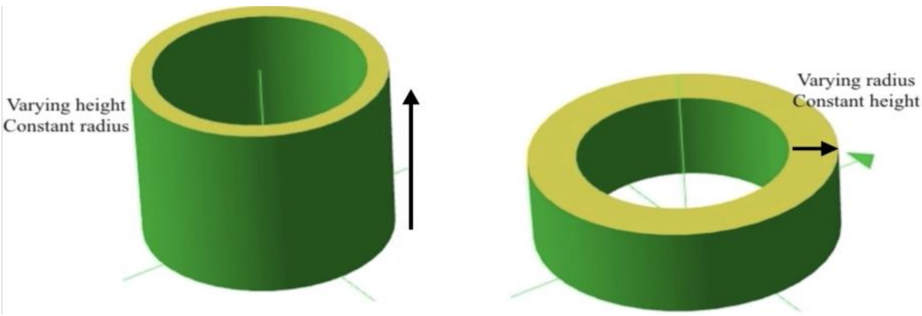


Figure 5. Two ways a cylinder can vary in volume

[4] I emphasize “equal in value” because the unit in each rate of change is, for example, $in^3/in$ (inch cubed per inch) whereas the value of surface area is in the unit $in^2$ (square inch). They are different quantities but have the same numerical values.

We "fill" the shell by letting cylinders vary over Δ*x* or Δ*y* intervals depending on which way we fill the shell. Figure 6 shows the shell being filled with cylinders that vary in radius over Δ*x*-intervals of size 0.5. The blue cylinder is meant to highlight that it is the cylinder varying in radius, so its rate of change of volume with respect to radius is the filling volume's rate of change. All accumulation up to that moment is fixed. It will not vary, so the cylinder's rate of change of volume with respect to *x* is the accumulated volume's rate of change with respect to *x*.

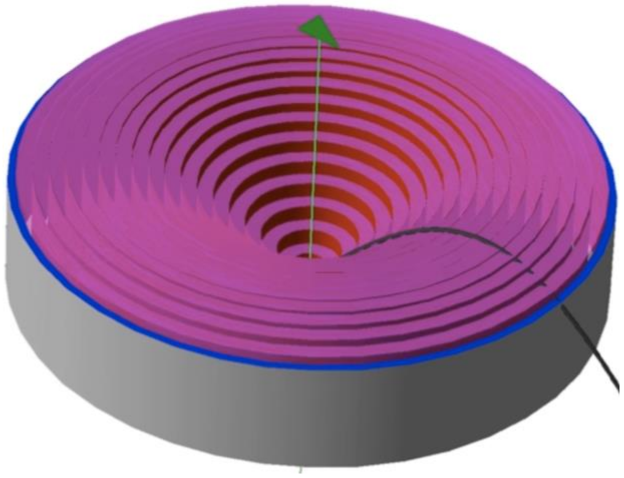

Figure 6. Filling the shell with cylinders of fixed height and varying radius. Imagine cylinders spreading outward from the center, the outermost cylinder always in blue

When Δ*x* is small enough, each cylinder varies in volume at the essentially constant rate of $2\pi x sin(x)\ in^3/in$ over its respective Δ*x*-interval, so when we use an infinitesimal value of Δ*x*, the volume of the enclosed region is $V(x) = \int_0^x 2\pi t sin(t) dt$ for any value of *x* from 0 to π, and therefore $V(\pi) = \int_0^\pi 2\pi t sin(t) dt$.

I and two colleagues (Mark Ashbrook, Fabio Milner) received a grant[5] from the U.S. National Science Foundation to implement these ideas in a curriculum that would

[5] Project DIRACC—Developing and Investigating a Rigorous Approach to Conceptual Calculus. NSF Grant No. DUE-1625678.

span the standard content of the first two semester-based calculus courses (Thompson et al., 2013; Thompson et al., 2019). Part of this grant was to develop validated assessment instruments for both courses that examined students' understandings of central ideas in the calculus independently of the curriculum they used. In this development we piloted both instruments twice with hundreds of students. The results were that DIRACC students consistently did significantly better than students in standard calculus or engineering calculus in understanding rate of change in relation to accumulation. However, the relationship was nevertheless difficult for students regardless of the curriculum (DIRACC final report; Thompson & Dreyfus, 2016).

The assessments clarified one aspect of this difficulty. Students were not thinking of variables' variation productively. For them to understand that an accumulation's rate of change at a value of its independent variable is the rate of change of accumulation over *the current* interval of variation requires that they think of any amount as having accumulated in bits and the accumulation having a rate of change within each of these bits. We designed one item (Figure 7) to get at their thinking with regard to this connection between accumulation and rate of change of accumulation.

> A car left from San Diego heading to New York. The car's average speed for the first 4 hours of the trip was 52 mph. In the next 0.003 hours, the car had an average speed of 71 mph. Which is the best estimate of how fast the car's distance from San Diego was changing at 4 hours after leaving San Diego?

Figure 7. Item from DIRACC Calculus 1 assessment

Options presented to students are below. Comments [in brackets] are explanations to you.

a. 52 mph [the car's average speed over the first four hours]

b. 52.014 mph [the car's weighted average speed over 4.003 hours]

c. 61.5 mph [$(52 + 71)/2$]

d. 71 mph [the car's average speed over the 0.003 hours immediately after the four-hour period]

e. Cannot be determined [There is insufficient information to answer the question]

We designed each option according to ways of thinking we detected among students in interviews held during assessment development. Option (a) reflects thinking of "distance so far" as like a rubber band stretching. There is no image of distance accruing in increments, so they think of average speed as "gone this speed most of the time". Option (b) reflects thinking of the car's average speed over the entire trip (total distance traveled divided by 4.003). Option (c) reflects thinking of "average" as meaning "add and divide by the number of numbers". Option (d), the correct answer, reflects the understanding that the total distance driven changes at the current rate of change of distance with respect to time, which in this case is best approximated by the average speed over the most recent interval of 0.003 hours. In other words, if $d(x)$ represents the distance from San Diego with respect to the number of hours driven, then the current rate of change of distance from San Diego with respect to time after *x* hours is best approximated by $(d(x + .003) - d(x))/.003$, or in this case $(d(4 + .003) - d(4))/.003$, which the problem text says is 71 mi/hr: the car's average speed over the interval [4,4.003].

Answers from 380 Calculus I and Calculus II students are given in Table 1. It is noteworthy that (d), the answer that reflects understanding that the car's rate of change of distance from San Diego (or any reference position) with respect to elapsed time is the car's current rate of change of distance with respect to time, was the least common choice among these students.

Table 1 *380 Calculus I and Calculus II students' answers to SD-NY question*

| 52 mph | 52.014 mph | 61.5 mph | **71 mph** | Cannot determine | No Answer |
|---|---|---|---|---|---|
| 24.4% | 17.7% | 26.2% | 13.1% | 16.2% | 2.3% |

The conceptual difficulty of seeing rate of change and accumulation as two sides of a coin is not limited to students. At a recent research conference I participated in a group of college calculus instructors working to develop a calculus concept assessment. The position taken by more than a few participants was that this item should be discarded because it was so obvious that the answer is 52 mph.

## Thoughts on relationships between mathematics and mathematics education

I mentioned my reliance on Harel's *Necessity Principle* in designing DIRACC's approach to motivating the ideas of derivative and antiderivative. There is a backstory to this decision. In 2011, when DIRACC was just a glimmer in my eye, I dedicated myself to teaching introductory calculus with a conceptual orientation using a standard textbook (Briggs, Cochran, & Gillet, 2011). I knew from prior research that robust understandings of constant and average rate of change were central to students' coherent understanding of calculus, so I emphasized these ideas early in order to leverage them throughout the course. However, the result was disappointing. Far too many students thought of derivatives and integrals in complete isolation. It was one afternoon that I said to my wife and colleague, Marilyn Carlson, "Oh! The only reason rate of change was necessary to my students was because I insisted they understand it!" There was no intellectual necessity to the idea. It was then that I realized that the only way that rate of change could be necessitated in Harel's sense was to *begin* with accumulation. Part of this realization was that I could ask students to use their ready-athand way of thinking of rate (i.e., $d = rt$) in developing the idea of accumulation from rate of change, but extend their understanding of rate of change when addressing the question of how we might derive a rate of change of accumulation from an accumulation function expressed in closed form, thus necessitating the idea of average rate of change over an interval as that constant rate of change that would produce the same change as original over that interval of change.

My perspective on the relationship between mathematics and mathematics education aligns closely with Harel's. I strive to see the mathematical potential in students' reasoning and envision a trajectory that might emerge with proper experiences upon which students might reflect. Such a trajectory necessarily included teachers who interact with students thoughtfully and caringly. To help teachers learn to interact with students to foster their mathematical development required we attended to teachers' in-the-moment thinking in the same way we attended to students' in-the-moment thinking and the mathematical meanings they hold as personal goals of instruction (e.g., Byerley & Thompson, 2017; Thompson, 2013, 2016; Thompson & Thompson, 1994, 1996; Yoon & Thompson, 2020) and incorporate results of such studies into mathematics teacher preparation programs and professional development programs for teachers.

I should note that while many students appreciated DIRACC's attempt to support their development of a coherent calculus, many others disliked the approach immensely. We systematically surveyed students' thoughts and opinions in each experimental implementation of the course. Comments such as, "I took calculus in high school. This is not calculus!" and "The textbook never explains anything." abounded. One student even wrote, "This is not mathematics. This is thinking!" Interviews with students made it clear that their image of mathematics teaching was that the teacher should demonstrate what students should be able to do. Their approach to homework was to not *read* the text, but instead to go to the assigned homework problems then look for similar examples in the textbook. This behavior continued despite explicit exhortation by instructors that it would be counterproductive, who also gave recommendations about how to use the textbook and how to approach problems so that students could solve them by drawing on meanings they had already developed.

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